Sextic K3 Surfaces with Geometric Picard Number 1

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Abstract

We give an explicit example of a K3 surface of degree 6 defined over the rational numbers with geometric Picard number 1. Previous work has shown that this is the generic case for complete intersections of K3 surfaces [3], but no explicit examples have been found in degree 6. Explicit examples have been found for the degree 2 [1] and degree 4 cases [4], and we extended the technique used to the case of K3 surfaces of degree 6.

Introduction

First, we define the geometric Picard number in a relatively general setting. Let X be a nonsingular variety over a field k.

- A **prime divisor** on X is a closed integral subvariety Y of codimension 1.
- A **(Weil) divisor** is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.
- a **principal divisor** is an element of $\operatorname{Div} X$ given by
$$(f) = \sum v_Y(f) \cdot Y$$
for some rational function f on X, where $v_Y(f)$ denotes the order of vanishing of f along Y.
- Two divisors D and D' are **linearly equivalent** if $D - D'$ is a principal divisor.
- The **Picard group** $\operatorname{Pic}(X)$ is the group of divisors on X modulo linear equivalence.
- The **geometric Picard number** of X is the rank of $\operatorname{Pic}(\overline{X})$, where $\overline{X} = X \times_{\operatorname{Spec}(k)} \operatorname{Spec} \overline{k}$.

Next, we define K3 surfaces and recount some basic properties of their Picard groups.

- A **K3 surface** is a smooth, projective, geometrically integral surface X over a field k whose canonical sheaf ω_X on X is trivial and $H^1(X, \mathcal{O}_X) = 0$.
- For a K3 surface X, $\operatorname{Pic}(X) \cong \mathbb{Z}^{\oplus \rho(X)}$ where $1 \leq \rho(X) \leq 22$. Note that the geometric Picard number of X is just the Picard number of $\overline{X}$, so it satisfies the same bound [2, p. 397].
- The geometric Picard number of a K3 surface over $\mathbb{F}_p$, is always even [2, Corollary 17.2.9]. Thus, the minimum geometric Picard number of a K3 surface over a finite field is 2.
- Reducing a K3 surface X defined over $\mathbb{Q}$ mod p induces a **specialization homomorphism** $\overline{\operatorname{sp}} : \operatorname{Pic}(X) \to \operatorname{Pic}(X_p)$ which is injective and compatible with the intersection products on X and X_p [2, Proposition 2.10].

To find the explicit example of a sextic K3 surface with geometric Picard number 1, we start by picking a sextic K3 surface $X_p \subset \mathbb{P}^4$ that contains a line for some prime p. We can project from this line, and because the line is contained in the surface we show that the projection is a degree 2 cover $X_p \to \mathbb{P}_2$, allowing us to use a preexisting algorithm for computing the Weil polynomial of degree 2 K3 Surfaces. We use the Weil polynomial to prove that $\operatorname{rk}(\operatorname{Pic}(\overline{X_p})) \leq 2$. Next, we find a lift of X_p to $\mathbb{Q}$, say X, which no longer contains the line L, and in fact does not contain any line. We prove that any lift of a line must be a line, meaning that the divisor class L in X_p does not lift, so $\operatorname{rk}(\operatorname{Pic}(\overline{X})) \leq 1$, which is its minimum value, so $\operatorname{rk}(\operatorname{Pic}(\overline{X})) = 1$.

Realizing X_p as a Degree 2 Surface

We start with a sextic K3 surface $X_p \subset \mathbb{P}^4$ over $\mathbb{F}_p$ defined by the vanishing of homogenous degree 2 and 3 polynomials f_2 and f_3, respectively. Further, we require that X_p contains the line L, which amounts to f_2 and f_3 being in the ideal of L.

To utilize the algorithms developed for degree 2 K3 surfaces, we need to view X_p as a degree 2 K3 surface. Projecting the surface X_p from the line L means mapping each point on X_p to the plane which meets both L and that point. The planes through L are parameterized by $\mathbb{P}^2$, so this gives a rational map to $\mathbb{P}^2$. Note that this map is not defined at L itself. To resolve the rational map, we blowup X at L, getting a morphism $f : X_p \to \mathbb{P}^2$.

We prove the following theorem, which implies that f is a degree 2 model for X_p.

Theorem 1. *Let $X_p \subset \mathbb{P}^4_{\mathbb{F}_p}$ be a sextic K3 surface given by the vanishing of homogenous degree 2 and degree 3 polynomials f_2 and f_3, respectively, and assume that X_p contains a line L. Then the projection of X from L is a finite, flat degree 2 morphism $X_p \to \mathbb{P}^2_{\mathbb{F}_p}$ whose branch locus is a sextic curve given by the vanishing of a single degree 6 polynomial f_6.*

Bounding the Picard Number of X_p using the Weil Polynomial

Let f be the Frobenius morphism on X_p, and let f^* be its pullback on $H^2_{et}(\overline{X}_p, \mathbb{Q}_l)$. We call the characteristic polynomial of f^* the **Weil polynomial** of X_p.

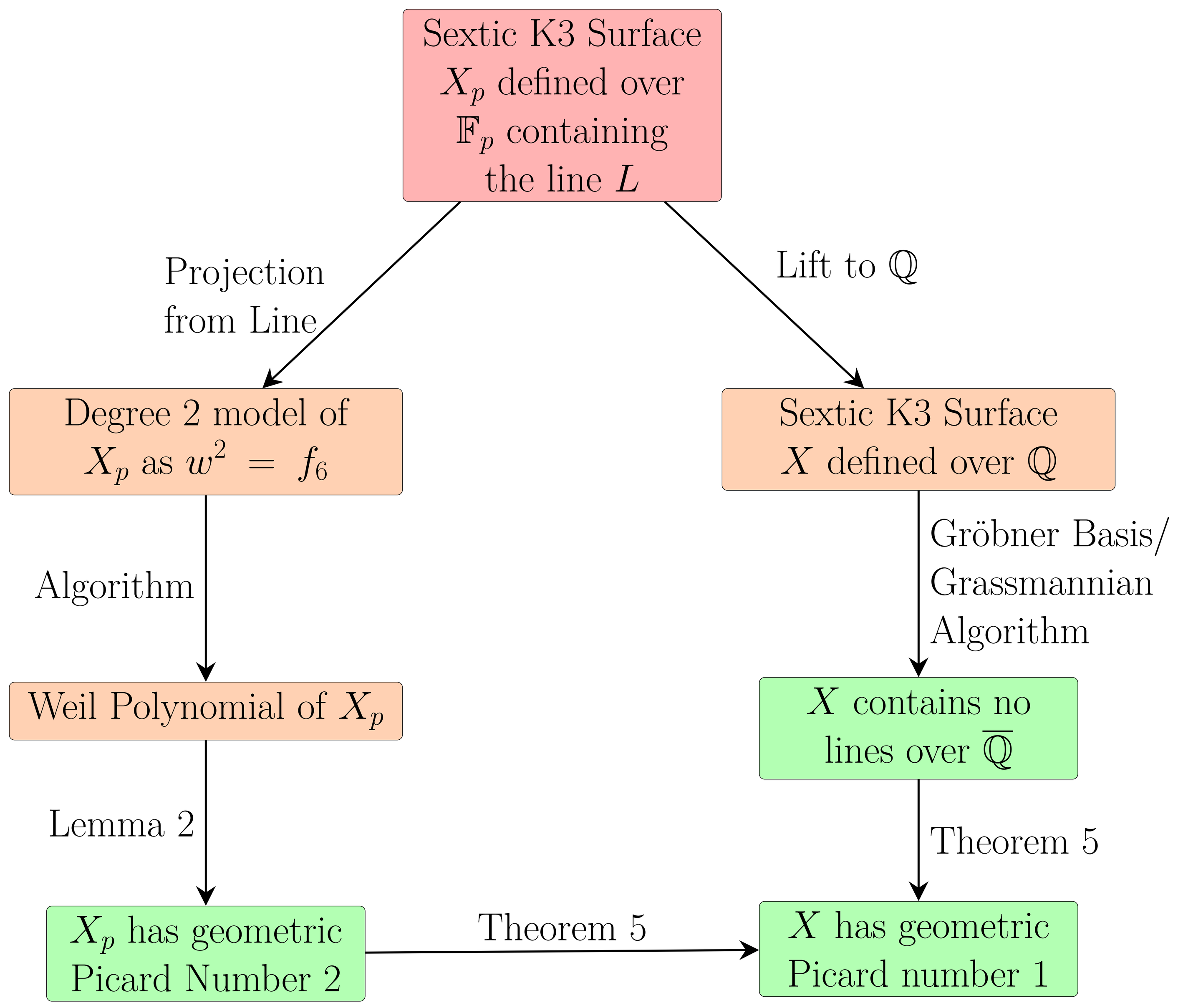

Figure 1: Summary of Proof Strategy, all algorithms in Magma

We have that $\operatorname{Pic}(\overline{X_p})$ injects into $H^2_{et}(\overline{X}, \mathbb{Q}_l)(1)$, and this injection respects the Galois action of $G(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, in particular the action of Frobenius [4].

By observing that all divisor classes are defined in some finite extension of $\mathbb{F}_p$, we have that some power of the Frobenius acts as the identity on $\operatorname{Pic}(\overline{X_p})$, so all eigenvalues of Frobenius on $\operatorname{Pic}(\overline{X_p})$ are roots of unity. If we let $f^*(1)$ be the induced automorphism on $H^2_{et}(\overline{X}, \mathbb{Q}_l)(1)$, we then know that $\operatorname{rk}\operatorname{Pic}(\overline{X_p})$ is bounded above by the number of eigenvalues of $f^*(1)$ that are roots of unity, counted with multiplicity. Eigenvalues of $f^*(1)$ differ from eigenvalues of f^* by a factor of p, so we have the following lemma:

Lemma 2 ([4, Corollary 2.3]). *The rank of $\operatorname{Pic}(\overline{X_p})$ is bounded from above by the number of eigenvalues λ of f^* for which λ/p is a root of unity, counted with multiplicity.*

We can use a preexisting algorithm in Magma to compute the Weil polynomial of X_p, so we randomly generate the defining equations f_2 and f_3 until we get a surface X_p such that the Weil polynomial has 2 roots which are roots of unity. This guarantees that $\operatorname{rk}\operatorname{Pic}(\overline{X_p}) \leq 2$.

Verifying that X contains no lines over $\overline{\mathbb{Q}}$

Now that we have bounded the geometric Picard number of X_p, we need to find a lift to a sextic K3 surface X over $\mathbb{Q}$ which contains no lines over $\overline{Q}$. To do this, we add 2 terms to f_2 which are not in the ideal of L but vanish mod p.

Next, we check that this lift contains no lines. Let $V = \mathbb{Q}^5$, and note that a line in $\mathbb{P}^4_{\mathbb{Q}}$ corresponds to a plane in V. Thus, the lines in $\mathbb{P}^4_{\mathbb{Q}}$ are parameterized by the Grassmannian $\operatorname{Gr}(2, 5)$ of 2-dimensional subspaces of 5-dimensional space.

We computed an explicit set of affine charts for $\operatorname{Gr}(2, 5)$ using Schubert cells. Because $\dim \operatorname{Gr}(2, 5) = 6$, each chart is the span of two vectors v_1 and v_2 parameterized by $y_1, ..., y_6$. Let C be the homogenous coordinate ring of $\mathbb{P}^4$ with homogenous coordinates $x_0, ..., x_4$, $R = \mathbb{Q}[y_1, ..., y_6]$, and $S = R[u_1, u_2]$.

Thus, if $f \in C$, then f is in the ideal of the line spanned by $v_1, v_2 \in V$ if and only if $f(u_1v_1 + u_2v_2) = 0$ for all $u_1, u_2 \in \mathbb{Q}$. If we expand the expressions $f_2(u_1v_1 + u_2v_2)$ and $f_3(u_1v_1 + u_2v_2)$, we can view them as polynomials in u_1 and u_2. These polynomials are zero for all u_1 and u_2 if and only if all their coefficients (taking u_1 and u_2 as unknowns and all other variables as constants) are all zero. The set of $y_1, ..., y_6$ which satisfy these constraints are an algebraic variety in $\mathbb{A}^6_{\mathbb{Q}}$ because each coefficient is a polynomial in $y_1, ..., y_6$.

In Magma, we can easily compute the ideal generated by all of these coefficients using a Gröbner basis calculation and check that it is the unit ideal, meaning that there are no such $y_1, ..., y_6$ over $\mathbb{Q}$ or in fact any algebraic extension of $\mathbb{Q}$, in particular $\overline{\mathbb{Q}}$.

Geometric Picard Number of X

We prove the following lemma about the behavior of the Picard group under specialization:

Lemma 3. *Given a K3 surface $X \subset \mathbb{P}^n_{\mathbb{Q}}$ such that $\overline{X}_p$ contains a line L and there exists $D \in \operatorname{Pic}(\overline{X})$ such that $\overline{\operatorname{sp}} = L$, then D is the class of a line.*

Additionally, Elsenhans and Jahnel showed the following lemma:

Lemma 4 ([1, Corollary 3.7]). *Let $p \neq 2$ be a prime number and X be a scheme proper and flat over $\mathbb{Z}$. Suppose that the special fiber X_p is nonsingular. Then, the cokernel of the specialization homomorphism $\overline{\operatorname{sp}} : \operatorname{Pic}(\overline{X}) \to \operatorname{Pic}(X_{\overline{\mathbb{F}}_p})$ is torsion free.*

With lemmas 3 and 4, we can prove the following theorem which allows us to conclude that X has geometric Picard number 1:

Theorem 5. *If $X \subset \mathbb{P}^n_{\mathbb{Q}}$ is a K3 surface containing no lines over $\overline{\mathbb{Q}}$ and the reduction X_p contains a line and has geometric Picard number 2 for some prime p of good reduction, then X has geometric Picard number 1.*

Proof. Let $H \in \operatorname{Pic}(\overline{X})$ be the hyperplane section. Assume to get a contradiction that X has geometric Picard number at least 2. If $\operatorname{rk}\operatorname{Pic}(\overline{X}) > 2$, we have a contradiction because $\overline{\operatorname{sp}}$ is an injective homomorphism from $\operatorname{Pic}(\overline{X})$ to $\operatorname{Pic}(\overline{X}_p)$, and $\operatorname{rk}\operatorname{Pic}(\overline{X}_p) = 2$. Thus, $\operatorname{rk}\operatorname{Pic}(\overline{X}) = 2$. Because $\overline{\operatorname{sp}}$ is injective, the image of $\overline{\operatorname{sp}}$ has rank 2 as well, so the cokernel has rank 0. However, by Lemma 4 we know that the cokernel is torsion free, so it must be 0, so $\overline{\operatorname{sp}}$ is surjective. Let L be the line in $\overline{X}_p$. Because $\overline{\operatorname{sp}}$ is surjective, we have a divisor class $E \in \operatorname{Pic}(\overline{X})$ such that $\overline{\operatorname{sp}}(E) = L$, but by Lemma 3 we have that E is the class of a line, a contradiction. □

Main Theorem

Theorem 6. *Let $X = V(f_2, f_3) \subset \mathbb{P}^4_{\mathbb{Q}}$ where*

$$\begin{aligned} f_2 = {} & x_0^2 - 3x_0x_1 + 3x_1^2 + 5x_0x_2 + 4x_1x_2 + 5x_2^2 - x_0x_3 - 2x_1x_3 - 3x_2x_3 - 5x_0x_4 + 5x_1x_4 \\ & + 47x_3^2 + 47x_4^2 \\ f_3 = {} & 2x_0^3 + 3x_0^2x_1 + 3x_0x_1^2 + x_1^3 - x_0x_1x_2 - 3x_1^2x_2 + 4x_0x_2^2 - 4x_1x_2^2 + 5x_2^3 + 4x_0^2x_3 + x_0x_1x_3 \\ & + 5x_1^2x_3 + 4x_0x_2x_3 + 4x_1x_2x_3 + -3x_2^2x_3 + 4x_1x_3^2 - x_2x_3^2 + 5x_0^2x_4 - 4x_0x_1x_4 + 2x_1^2x_4 \\ & + x_0x_2x_4 + 4x_1x_2x_4 - 2x_2^2x_4 + 4x_0x_3x_4 - 3x_2x_3x_4 - x_0x_4^2 + -x_1x_4^2 + 5x_2x_4^2, \end{aligned}$$

and $x_0, ..., x_4$ are the homogenous coordinates for $\mathbb{P}^4_{\mathbb{Q}}$. Then X is a sextic K3 surface with geometric Picard number 1.

Proof. Let $p = 47$, and note that the reduction of X mod p is defined by the same equations except for the removal of the last two terms of f_2. Note that at least one of x_0, x_1, or x_2 divides every remaining term, so X_p contains the line $L = V(x_0, x_1, x_2)$.

We computed that the Weil polynomial of X_p is

$$\begin{aligned} & (x - 47)^2(x^{20} + 35x^{19} + 1410x^{18} + 79524x^{17} - 311469x^{16} + 39037448x^{15} + 5504280168x^{14} \\ & - 86233722632x^{13} - 1013246240926x^{12} - 666716026529308x^{11} - 7833913311719369 0x^{10} \\ & - 1472775702603241372x^9 - 49443184301680246 06x^8 - 929531864871588625928x^7 \\ & + 131063992946893996255848x^6 + 2053335889501339274674952x^5 \\ & - 361900450524611047161 46029x^4 + 204111854095880630599063603 56x^3 \\ & + 7994380952088658031796657806 10x^2 + 43835855553952808207685006970115x \\ & + 27666687119623358094507480113 42401). \end{aligned}$$

It is easy to check that the second factor contains no cyclotomic factors, so the Weil polynomial has exactly 2 roots which are p times a root of unity. Thus, by Lemma 2, X_p has geometric Picard number at most 2.

Finally, we can check using the method described above that X contains no lines, so X has geometric Picard number 1 by Theorem 4. □

References

[1] Andreas-Stephan Elsenhans and Jörg Jahnel, *The picard group of a k3 surface and its reduction modulo p*, **5**, no. 8, 1027–1040. MR 2948470

[2] Daniel Huybrechts, *Lectures on k3 surfaces*, University Press.

[3] Tomohide Terasoma, *Complete intersections with middle picard number 1 defined over q*, **189**, no. 2, 289–296.

[4] Ronald van Luijk, *K3 surfaces with picard number one and infinitely many rational points*, **1**, no. 1, 1–15.