Consecutive patterns in permutations: clusters, generating functions and asymptotics

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February 2012

Consecutive patterns

$\pi = \pi_1\pi_2 \dots \pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_m.$

Classical definition:

π *contains* σ if it has a subsequence order-isomorphic to σ.

Ex: 25134 contains 132

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π *contains* σ *as a consecutive pattern* if it has a subsequence of adjacent entries order-isomorphic to σ.

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Ex: 25134 avoids 132, but 42531 contains 132
15243 contains two occurrences of 132

In this talk, containment and avoidance will always refer to consecutive patterns.

History

Consecutive patterns appear naturally in combinatorics:

- Occurrences of 21 are *descents*.
- Occurrences of 132 and 231 are *peaks*.
- Permutations avoiding 123 and 321 are *alternating permutations*.

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- Occurrences of 132 and 231 are *peaks*.
- Permutations avoiding 123 and 321 are *alternating permutations*.

The systematic study of consecutive patterns in permutations started 12 years ago.

Work in the area by Noy, Babson, Steingrímsson, Claesson, Mansour, Kitaev, Mendes, Remmel, Dotsenko, Khoroshkin, Shapiro, Ehrenborg, Perry, Baxter, Nakamura, Zeilberger among others.

Some of the questions being studied

Let $c_\sigma(\pi) = \#$ of occurrences of σ in π,
$\alpha_n(\sigma) = \#$ of permutations of length n that avoid σ,

$$P_\sigma(u,z) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} u^{c_\sigma(\pi)}\frac{z^n}{n!}, \qquad P_\sigma(0,z) = \sum_{n\geq 0}\alpha_n(\sigma)\frac{z^n}{n!}.$$

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- Exact enumeration: find $P_\sigma(u,z)$ or $P_\sigma(0,z)$.
- Classification of patterns according to Wilf-equivalence.
  We write $\sigma \sim \tau$ if $P_\sigma(u,z) = P_\tau(u,z)$.
- Asymptotic behavior of $\alpha_n(\sigma)$.
  Comparison of $\alpha_n(\sigma)$ for different patterns.

Patterns of small length

Length 3: two classes (compare to one class in classical case)

$123 \sim 321$

$132 \sim 231 \sim 312 \sim 213$

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$123 \sim 321$
$132 \sim 231 \sim 312 \sim 213$

Length 4: seven classes (compare to three classes in classical case)

$1234 \sim 4321$	enumeration solved
$2413 \sim 3142$	enumeration unsolved
$2143 \sim 3412$	
$1324 \sim 4231$	
$1423 \sim 3241 \sim 4132 \sim 2314$	
$1342 \sim 2431 \sim 4213 \sim 3124 \overset{*}{\sim} 1432 \sim 2341 \sim 4123 \sim 3214$	
$1243 \sim 3421 \sim 4312 \sim 2134$	

All $\sim$ follow from reversal and complementation except for $\overset{*}{\sim}$.

Clusters

We use an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

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More precisely, a *k*-cluster is $(\pi; i_1, i_2, \dots, i_k)$ where

- $\pi \in \mathcal{S}_n$,
- $1 = i_1 < i_2 < \dots < i_k = n-m+1$,
- $\pi_{i_j}\pi_{i_j+1}\dots\pi_{i_j+m-1}$ is an occurrence of σ for all j,
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Ex: $(142536879; 1, 3, 6)$ is a 3-cluster w.r.t. to 1324.

Set of overlaps

$O_\sigma = \{i : \sigma_{i+1}\sigma_{i+2}\ldots\sigma_m$ and $\sigma_1\sigma_2\ldots\sigma_{m-i}$ are the same 'pattern'$\}$

(i.e., overlapping occurrences of σ may be shifted by i positions)

Ex: $O_{1324} = \{2, 3\}$, $O_{15243} = \{3, 5\}$, $O_{12\ldots m} = \{1, 2, \ldots, m-1\}$.

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In a cluster $(\pi; i_1, i_2, \dots, i_k)$ w.r.t. σ we have $i_{j+1} - i_j \in O_\sigma$ for all j.

Ex: $(142536879; 1, 3, 6)$ is a cluster w.r.t. 1324.

The cluster method

Let the EGF for clusters be

$$R_\sigma(t,z) = \sum_{n,k} r_{n,k}\, t^k \frac{z^n}{n!},$$

where $r_{n,k} :=$ number of k-clusters of length n w.r.t. σ.

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Theorem (Goulden-Jackson '79, adapted)

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$$P_\sigma(u,z) = \frac{1}{1-z-R_\sigma(u-1,z)}.$$

This reduces the computation of $P_\sigma(u,z)$ to the enumeration of clusters.

Clusters as linear extensions of posets

Let $\sigma \in \mathcal{S}_m$,
Let $1 = i_1 < i_2 < \cdots < i_k = n-m+1$ with $i_{j+1} - i_j \in O_\sigma$ for all j.

Then

$$(\pi; i_1, \ldots, i_k) \text{ is a cluster}$$
$$\Updownarrow$$
$$\pi_{i_j}\pi_{i_j+1}\ldots\pi_{i_j+m-1} \text{ is an occurrence of } \sigma \text{ for all } j$$

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Let $\sigma \in \mathcal{S}_m$, $\quad \varsigma = \sigma^{-1}$
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$$\Updownarrow$$
$$\pi_{\varsigma_1+i_j-1} < \pi_{\varsigma_2+i_j-1} < \cdots < \pi_{\varsigma_m+i_j-1} \text{ for all } j$$
$$\Updownarrow$$

π is a linear extension of the poset defined by these relations
(we call this a cluster poset)

Clusters as linear extensions of posets: example

Take $\sigma = 14253$. Then $O_\sigma = \{2, 4\}$, $\sigma^{-1} = 13524$.

$$(\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6\pi_7\pi_8\pi_9\pi_{10}\pi_{11}; 1, 3, 7) \text{is a cluster}$$

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$$\pi_1 < \pi_3 < \pi_5 < \pi_2 < \pi_4$$
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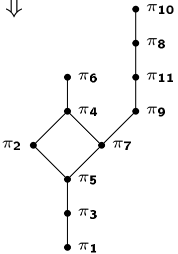

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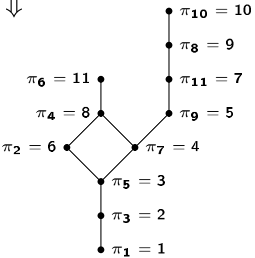

Ex: 1 6 2 8 3 11 4 9 5 10 7

Notation

$$P_\sigma(u,z)=\frac{1}{\omega_\sigma(u,z)}=\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} u^{c_\sigma(\pi)}\frac{z^n}{n!}$$ (EGF for occurrences of σ)

$$P_\sigma(0,z)=\sum_{n\geq 0}\alpha_n(\sigma)\frac{z^n}{n!}$$

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We will give differential equations for $\omega_\sigma(u,z)$ for some patterns σ.

- All derivatives will always be with respect to z.
- Initial conditions will be omitted.

The pattern $\sigma = 12\dots m$

Theorem (E.-Noy '01)

For $\sigma = 12\dots m$, $\omega_\sigma(u,z)$ *is the solution of*

$$\omega^{(m-1)} + (1-u)(\omega^{(m-2)} + \cdots + \omega' + \omega) = 0.$$

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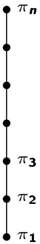
Ex:

$$P_{123}(0,z) = \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z + \frac{\pi}{6})}$$

$$P_{1234}(0,z) = \frac{2}{\cos z - \sin z + e^{-z}}$$

In general, $$\omega_{12\ldots m}(0,z) = \sum_{j\geq 0}\frac{z^{jm}}{(jm)!} - \sum_{j\geq 0}\frac{z^{jm+1}}{(jm+1)!}$$

Proof sketch



For each choice of $1 = i_1 < i_2 < \cdots < i_k = n-m+1$ with $i_{j+1} - i_j \in O_{12\ldots m} = \{1, 2, \ldots, m-1\}$ for all j,

there is exactly one cluster $(\pi; i_1, \ldots, i_k)$,

because the cluster posets are chains $\pi_1 < \pi_2 < \cdots < \pi_n$.

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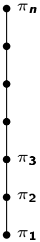

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We deduce that the EGF $R_{12\ldots m}(t, z)$ for clusters satisfies

$$R^{(m-1)} = t(R^{(m-2)} + \cdots + R' + R + z),$$

which gives the diff. eq. for $\omega_{12\ldots m}(u, z)$.

Chain patterns

We say that σ is a chain pattern if all the cluster posets are chains.

Theorem (E.-Noy '11)

Let $\sigma \in \mathcal{S}_m$ be a chain pattern. Then $\omega_\sigma(u,z)$ is the solution of

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Corollary

Let $\sigma = 123\dots(s-1)(s+1)s(s+2)(s+3)\dots m$. Then $\omega_\sigma(u,z)$ is the solution of

$$\omega^{(m-1)} + (1-u)(\omega^{(m-s-1)} + \dots + \omega' + \omega) = 0.$$

Examples

Ex: For $\sigma = 12435$, $\omega_\sigma(u,z)$ satisfies

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Ex: Both $\omega_{123546}(u,z)$ and $\omega_{124536}(u,z)$ satisfy

$$\omega^{(5)} + (1-u)(\omega' + \omega) = 0.$$

This proves Nakamura's conjecture that $123546 \sim 124536$.

Non-overlapping patterns

Recall: $\sigma \in \mathcal{S}_m$ non-overlapping if $O_\sigma = \{m-1\}$, i.e., two occurrences of σ can't overlap in more than one position.

Ex: 132, 1243, 1342, 34671285.

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Proposition (Dotsenko-Khoroshkin, Remmel '10)

For $\sigma \in \mathcal{S}_m$ non-overlapping, $P_\sigma(u,z)$ depends only on σ_1 and σ_m.

Non-overlapping patterns

Theorem (E.-Noy '01)

Let $\sigma \in \mathcal{S}_m$ *be non-overlapping with* $\sigma_1 = 1$, $\sigma_m = b$. *Then* $\omega_\sigma(u, z)$ *is the solution of*

$$\omega^{(b)} + (1-u)\frac{z^{m-b}}{(m-b)!}\omega' = 0.$$

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Ex: For $b = 2$,

$$\omega_\sigma(u, z) = 1 - \int_0^z e^{(u-1)\frac{v^{m-1}}{(m-1)!}} dv.$$

$$P_{132}(u, z) = \frac{1}{1 - \int_0^z e^{(u-1)t^2/2} dt}$$

Proof sketch using cluster method

Suppose $a = \sigma_1 < \sigma_m = b$.
Clusters $(\pi; i_1, i_2, \dots, i_k)$ w.r.t. σ satisfy
$i_{j+1} - i_j = m - 1$ for all j.

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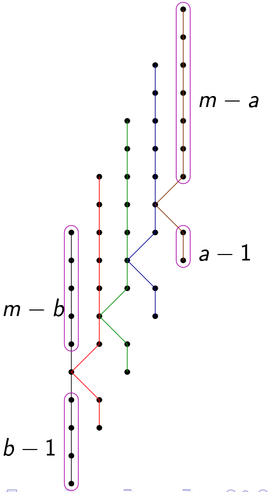
They are linear extensions of posets like this:



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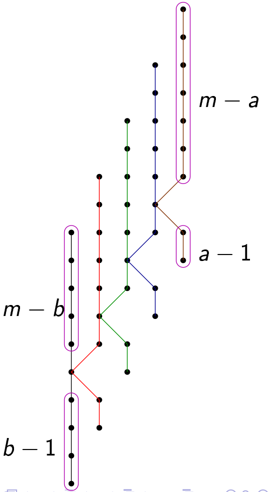


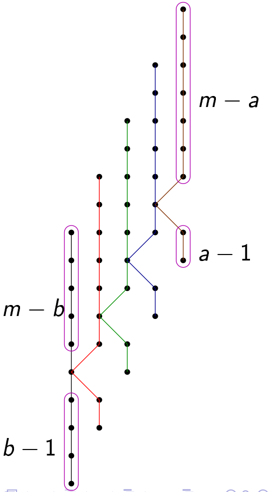
For $\sigma_1 = 1$, we deduce a diff. eq. for the EGF for clusters:

$$R^{(b)} = t\frac{z^{m-b}}{(m-b)!}(1 + R'),$$

which gives the diff. eq. for $\omega_\sigma(u, z)$.

Consequences of the proof

- These posets depend only on $a = \sigma_1$ and $b = \sigma_m$.



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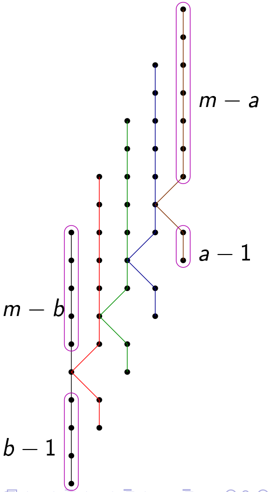
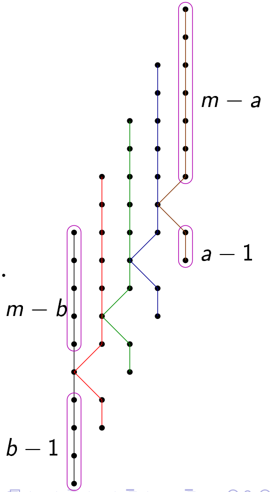
- k-clusters have size $n = k(m-1)+1$.



Consequences of the proof

- These posets depend only on $a = \sigma_1$ and $b = \sigma_m$.
- k-clusters have size $n = k(m-1)+1$.
- If d_k is the number of k-clusters, then

$$\omega_\sigma(u,z) = 1 - z - \sum_{k\geq 1} (u-1)^k d_k \frac{z^{k(m-1)+1}}{(k(m-1)+1)!}.$$



The patterns 12534 and 13254

Proposition (E.-Noy '11)

$\omega_{12534}(u,z)$ *is the solution of* $\omega^{(4)} + (1-u)z(\omega'' + \omega') = 0$,

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Wilf-equivalence classes

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There are

- 2 classes for length 3,
- 7 classes for length 4,
- 25 classes for length 5,
- 92 classes for length 6.

The pattern 1324

Theorem (E.-Noy '11)

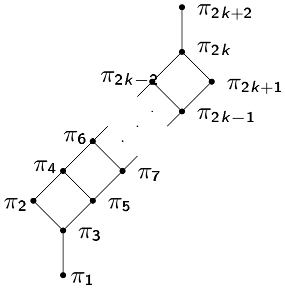
For $\sigma = 1324$, $\omega_\sigma(u,z)$ is the solution of

$$z\omega^{(5)} - ((u-1)z-3)\omega^{(4)} - 3(u-1)(2z+1)\omega^{(3)} + (u-1)((4u-5)z-6)\omega'' + (u-1)(8(u-1)z-3)\omega' + 4(u-1)^2 z\omega = 0$$

Proof sketch

Clusters $(\pi; i_1, \dots, i_k)$ satisfy
$i_{j+1} - i_j \in O_{1324} = \{2, 3\}$ for all j.

Clusters where $i_{j+1} - i_j = 2$ for all j correspond to linear extensions of



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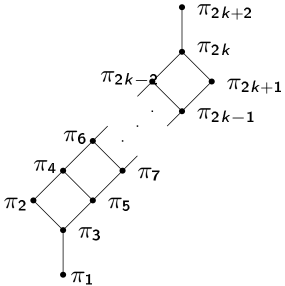

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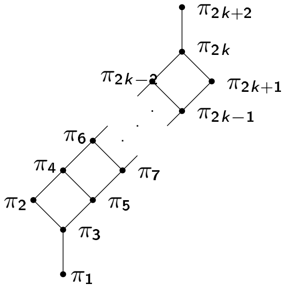


These are counted by the Catalan numbers.

For general clusters, we get towers of these posets. We deduce the OGF for clusters

$$\sum_{k,n} r_{n,k} t^k x^n = \frac{x(1 - 2tx(1+x) + \sqrt{1 - 4tx^2})}{2(1 - tx(1+x)^2)} - x.$$

We turn this into an diff. eq. for the OGF, and then into a diff eq. for the EGF $R_{1324}(t, z)$ and for $\omega_{1324}(u, z)$.

The pattern $134\dots(s+1)2(s+2)(s+3)\dots m$

Theorem (E.-Noy '11)

The OGF for clusters w.r.t. $\sigma = 134\dots(s+1)2(s+2)(s+3)\dots m$ *is*

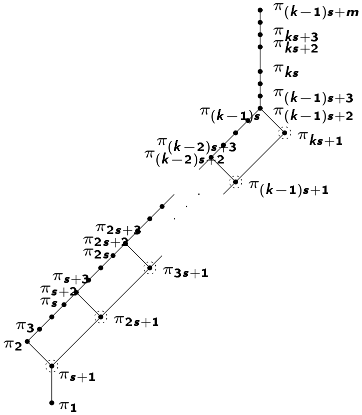
$$\sum_{k,n} r_{n,k} t^k x^n = \frac{x^{m-s}(B(tx^s)-1)}{1-(x+x^2+\cdots+x^{m-s-1})(B(tx^s)-1)},$$

where

$$B(x) = 1 + xB(x)^s.$$

For small s and m, we can deduce a differential equation for $\omega_\sigma(u,z)$.

Proof idea



The cluster posets are towers of posets like this one.

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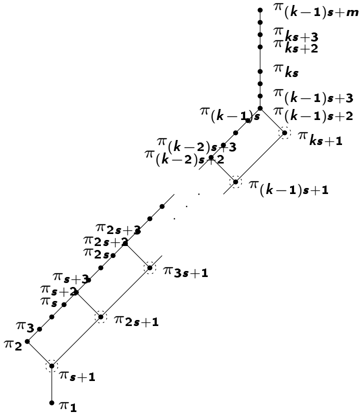

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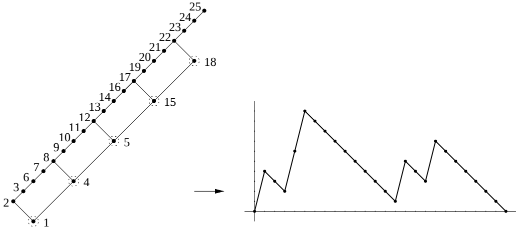
Linear extensions of this poset are in bijection with certain generalized Dyck paths, whose OGF is $B(x)$.



Other patterns of length 4

For the remaining cases, 1423, 2143 and 2413, we have recurrences for the cluster numbers, but no closed form or diff. eq. for $\omega_\sigma(u, z)$.

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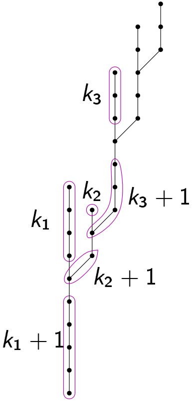
Equivalent to showing that $S(x)$ defined by

$$S(x) = 1 + \frac{x}{1+x} S\left(\frac{x}{1+x^2}\right) \quad \text{is not D-finite.}$$



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Theorem (E.-Noy '11)

Suppose that $\exists\alpha > 0$ s.t. all cluster posets w.r.t. σ of size n contain a chain of length $\geq \alpha n$. Then, for every fixed $u \in \mathbb{C}$, $\omega_\sigma(u, z)$ is an entire function of z.

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This applies to

- all chain patterns,
- all patterns with $\sigma_1 = 1$,
- all non-overlapping patterns.

Proof idea

Intuition: Posets containing a large chain have few linear extensions.

Bounding the number of linear extensions of the cluster posets,

$$\sum_k r_{n,k} \leq 2^n n^{n-\alpha n}$$

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$\Rightarrow$ $R_\sigma(t,z) = \sum r_{n,k} t^k \frac{z^n}{n!}$ is entire as a function of z (has ∞ radius of convergence)

$\Rightarrow$ $\omega_\sigma(u,z)$ is entire as a function of z

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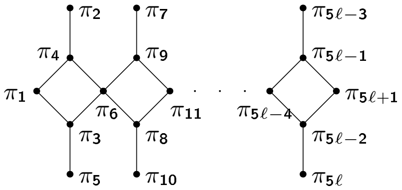
Intuition: some cluster posets have too many linear extensions.



Asymptotic behavior

Proposition (E. '05)

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$$\rho_\sigma := \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} \qquad \textit{exists.}$$

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For some σ, ρ_σ is known exactly. For other σ it can be estimated.

Theorem (Ehrenborg-Kitaev-Perry '11)

For every σ,

$$\frac{\alpha_n(\sigma)}{n!} = \gamma \rho^n + O(\delta^n),$$

for some constants γ and $\delta < \rho$.

The proof uses methods from spectral theory.

A conjecture

Conjecture (E.-Noy. '01)

For every $\sigma \in \mathcal{S}_m$ *there exists* n_0 *such that*

$$\alpha_n(\sigma) \leq \alpha_n(12\ldots m)$$

for all $n \geq n_0$.

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Theorem (E.-Noy. '11)

The above conjecture holds if σ is non-overlapping.

Proof sketch

Let $\sigma \in \mathcal{S}_m$ be non-overlapping. Want to show: $\rho_\sigma < \rho_{12\ldots m}$.

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Using singularity analysis and the fact that $\omega_\sigma(z)$ and $\omega_{12\ldots m}(z)$ are entire, we get

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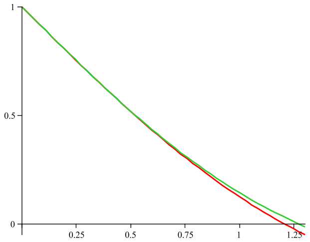

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Enough to show that $\omega_{12\dots m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$.

$$\omega_{12\dots m}(z) = \sum_{j\geq 0} \frac{z^{jm}}{(jm)!} - \sum_{j\geq 0} \frac{z^{jm+1}}{(jm+1)!} < 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m+1)!} + \frac{z^{2m}}{(2m)!},$$

$$\omega_\sigma(z) = 1 - z - \sum_{k\geq 1} (-1)^k d_k \frac{z^{k(m-1)+1}}{(k(m-1)+1)!} > 1 - z + \frac{z^m}{m!} - d_2 \frac{z^{2m-1}}{(2m-1)!}.$$

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Using $d_2 \leq \binom{2m-3}{m-2}$ and algebraic manipulations $\longrightarrow$ $\wedge$

$$\omega_\sigma(z) = 1-z-\sum_{k\geq 1}(-1)^k d_k \frac{z^{k(m-1)+1}}{(k(m-1)+1)!} > 1-z+\frac{z^m}{m!}-d_2\frac{z^{2m-1}}{(2m-1)!}.$$

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Last-minute update

Proved while preparing this talk:

- For every $\sigma \in \mathcal{S}_m$ there exists n_0 s.t.

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for all $n \geq n_0$. (This is the [E.-Noy. '01] conjecture.)

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- For every non-overlapping $\sigma \in \mathcal{S}_m$ there exists n_0 s.t.

$$\alpha_n(123\ldots(m-2)m(m-1)) \leq \alpha_n(\sigma) \leq \alpha_n(134\ldots m2)$$

for all $n \geq n_0$.

Open problems

- Find a proof of the Ehrenborg-Perry-Kitaev Theorem

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- Conjecture (Nakamura '11): For every $\sigma \in \mathcal{S}_m$ there exists n_0 s.t.

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Thank you