Schur-positive grid classes

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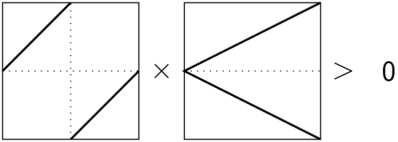
Joint work with Yuval Roichman



Combinatorics Graduate Student Conference, Apr 1-3, 2016

Clemson University

Pattern avoidance

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Let

$$\mathcal{S}_n(B) = \{\pi \in \mathcal{S}_n : \pi \text{ avoids } B\}.$$

Statistics on permutations

For $\pi \in \mathcal{S}_n$, define its

- descent set

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Example: For $\pi = 51432$,

$$\mathrm{Des}(\pi) = \{1, 3, 4\}, \quad \mathrm{inv}(\pi) = 4 + 2 + 1 = 7.$$

Standard Young tableaux

$\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_1 + \lambda_2 + \cdots = n$. We write $\lambda \vdash n$.

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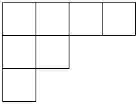

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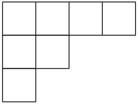

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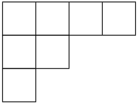

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$\mathrm{Des}(T) = \{2, 4\}$

Its descent set is $\mathrm{Des}(T) = \{i : \ i+1 \text{ is in a lower row than } i\}$.

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Symmetric functions

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$$g = 2\sum_i x_i^2 + \sum_{i<j} x_i x_j = 2x_1^2 + 2x_2^2 + \dots + x_1 x_2 + x_1 x_3 + \dots$$

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The set of homogeneous symmetric functions of degree k forms a vector space over $\mathbb{Q}$, denoted by Λ_k.

Schur functions

For $\lambda \vdash k$, define the Schur function

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \prod_i x_i^{\text{number of } i\text{s in } T}.$$

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$$s_{2,1} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3 + \dots$$

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Theorem

Schur functions are symmetric, and $\{s_\lambda : \lambda \vdash k\}$ is a basis for Λ_k.

Schur-positivity

A symmetric function is Schur-positive if all the coefficients in its expansion in the Schur basis are nonnegative.

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The Littlewood–Richardson rule gives a combinatorial interpretation of the coefficients $c^\nu_{\lambda,\mu}$, showing that $s_\lambda s_\mu$ is Schur-positive.

Quasi-symmetric functions

A quasi-symmetric function is a formal power series $f(x_1, x_2, \dots)$ of bounded degree where, for every fixed $\alpha_1, \dots, \alpha_k$, the coefficient of $x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$ is the same for any increasing indices $i_1 < \cdots < i_k$.

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For $\pi \in \mathcal{S}_n$, define the quasisymmetric function

$$F_\pi = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(\pi)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

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Example: $\pi = 132$, $\mathrm{Des}(\pi) = \{2\}$.

$$F_{132} = x_1 x_1 x_2 + x_1 x_1 x_3 + x_1 x_2 x_3 + x_2 x_2 x_3 + \ldots.$$

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We define $\mathcal{Q}(A)$ similarly if A is a multiset.

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Theorem (Gessel '84)

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For $J \subseteq \{1, \ldots, n-1\}$, define the inverse descent class

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Corollary

D_J^{-1} is Schur-positive.

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Theorem (Adin, Roichman '15)

For every k, the set $\{\pi \in \mathcal{S}_n : \mathrm{inv}(\pi) = k\}$ is Schur-positive.

Arc permutations

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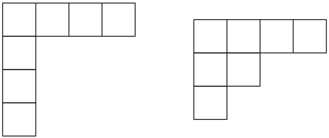

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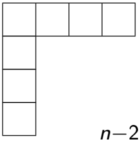

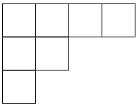

Corollary

$\mathcal{A}_n$ is Schur-positive and

$$\mathcal{Q}(\mathcal{A}_n) = s_n + s_{1^n} + \sum_{k=2}^{n-2} s_{n-k,2,1^{k-2}} + 2\sum_{k=1}^{n-2} s_{n-k,1^k}.$$

Geometric grid classes

For a $\{0, 1, -1\}$-matrix M, let $\Gamma(M)$ be the set of line segments of slope ± 1 whose locations are determined by the entries of M.

Geometric grid classes

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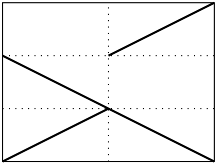
$\Gamma(M) =$ 

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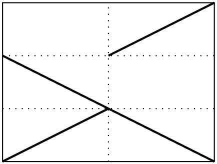

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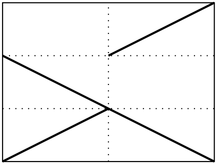

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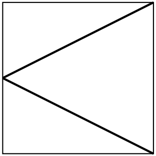

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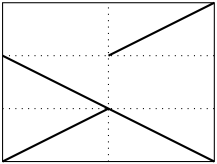

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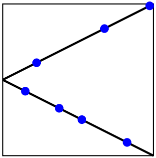

Grid classes and pattern avoidance

Theorem (Albert, Atkinson, Bouvel, Ruškuc, Vatter '13)

Every geometric grid class can be characterized by avoidance of a finite set of patterns.

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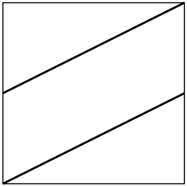
Example:

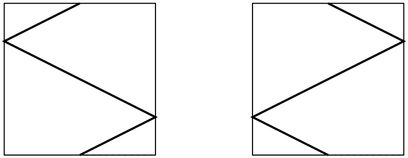
$$\mathcal{G}_n\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathcal{S}_n(321, 2143, 2413).$$



Arc permutations as grid classes

Arc permutations can be expressed as a union of two (geometric) grid classes:

$$\mathcal{A}_n = \mathcal{G}_n \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \cup \mathcal{G}_n \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}.$$



Elementary examples of Schur-positive grid classes

One-Column grid classes

Proposition

Every one-column grid class is Schur-positive.

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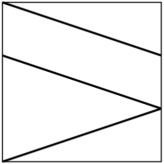
Example:

Let $\mathcal{H}_n = \mathcal{G}_n \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$



$$\mathcal{Q}(\mathcal{H}_5) = s_5 + 2\,s_{4,1} + 2\,s_{3,2} + 3\,s_{3,1,1} + 4\,s_{2,2,1} + 4\,s_{2,1,1,1} + s_{1,1,1,1,1}$$

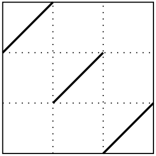
Elementary examples

Co-layered permutations

Let $\mathcal{L}_n^k$ be the grid class determined by the $k \times k$ identity matrix.

Example:

$\mathcal{L}_n^k = \mathcal{G}_n(\mathrm{Id}_3)$



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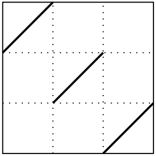

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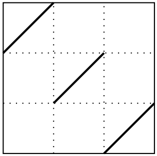

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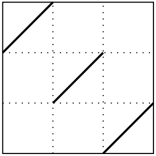

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Main theorem

Given $A, B \in \mathcal{S}_n$, let AB be the multiset of permutations obtained as products $\pi\sigma$ where $\pi \in A$ and $\sigma \in B$.

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In fact,

$$\mathcal{Q}(AD_J^{-1}) = \mathcal{Q}(A) * \mathcal{Q}(D_J^{-1}),$$

where $*$ denotes the Kronecker product.

Application: vertical rotations

Let $c \in \mathcal{S}_n$ be the n-cycle $c = (1, 2, \dots, n)$, and let $C_n = \langle c \rangle = \{c^k : \ 0 \leq k < n\}$ be the subgroup it generates.

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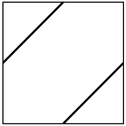

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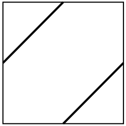

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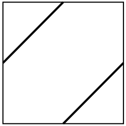

Corollary

- *For $J \subseteq [n-1]$, the multiset $C_n D_J^{-1}$ is Schur-positive.*
- *For a one-column grid class $\mathcal{H}_n$, the multiset $C_n \mathcal{H}_n$ is Schur-positive.*

Arc permutations revisited

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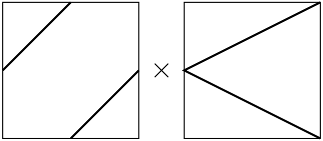

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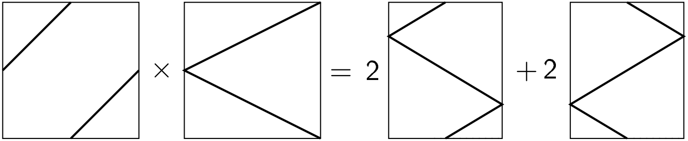

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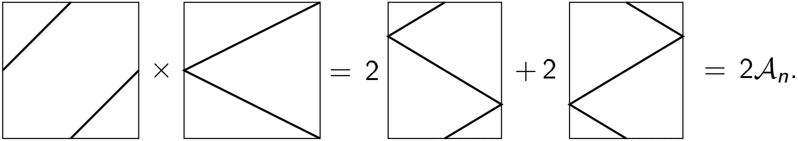

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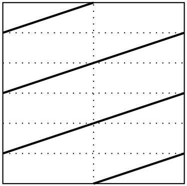
In fact, $\mathcal{Q}(AC_n) = \mathcal{Q}(A)s_1$.

Equivalently, if A "corresponds" to an $\mathcal{S}_{n-1}$-representation ρ, then AC_n "corresponds" to the $\mathcal{S}_n$-representation $\rho\uparrow^{\mathcal{S}_n}$.

Horizontal rotations

Let M_k be the $2k \times 2$ matrix whose odd rows are $(1, 0)$ and whose even rows are $(0, 1)$.

$\mathcal{G}_n(M_3)$



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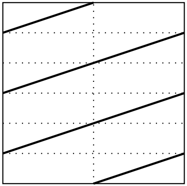
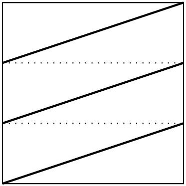
$\mathcal{G}_n(M_3)$  $\mathcal{G}_n\begin{pmatrix}1\\1\\1\end{pmatrix}$ 

Corollary

$\mathcal{Q}(\mathcal{G}_n(M_k))$ is Schur-positive for all k.

Stacking operations

Given matrices M_1 and M_2, one of which has one column, let $\Gamma\binom{M_1}{M_2}$ be the grid obtained by stacking $\Gamma(M_1)$ atop $\Gamma(M_2)$, and $\mathcal{G}_n\binom{M_1}{M_2}$ the corresponding grid class.

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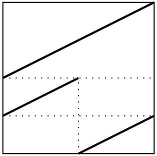
Example

$$\mathcal{G}_n\begin{pmatrix} & 1 & \\ 1 & & 0 \\ 0 & & 1 \end{pmatrix}$$



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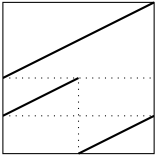

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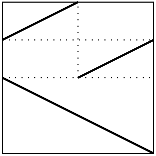

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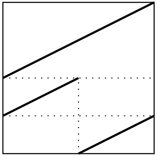

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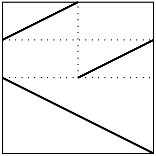

Proposition The above two grids are Schur-positive.

Question: If M_1 has one column and $\mathcal{G}(M_2)$ is Schur-positive, is $\mathcal{G}_n\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ necessarily Schur-positive?

Open questions

Conjecture

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Question: Which pairs of Knuth classes $A, B \subseteq \mathcal{S}_n$ satisfy $\mathcal{Q}(AB) = \mathcal{Q}(A) * \mathcal{Q}(B)$?

Thanks