Rowmotion on 321-avoiding permutations

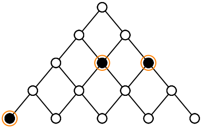
Sergi Elizalde
(joint work with Ben Adenbaum)

Dartmouth College

BIRS Dynamical Algebraic Combinatorics
November 2021

Rowmotion on antichains and order ideals

antichains



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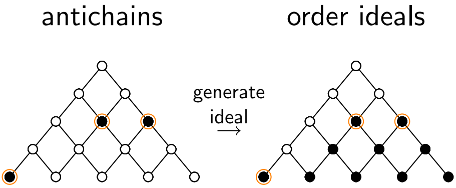

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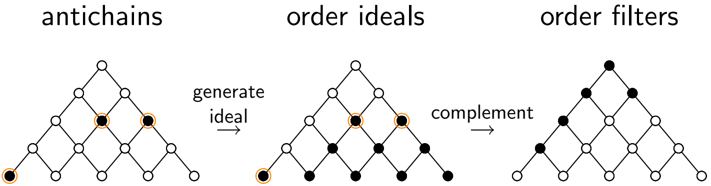

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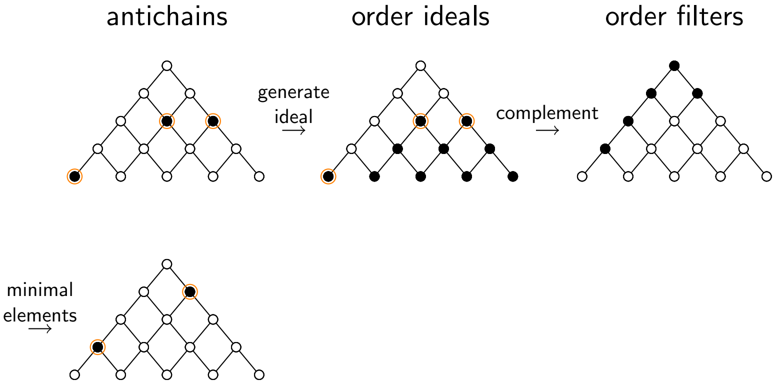

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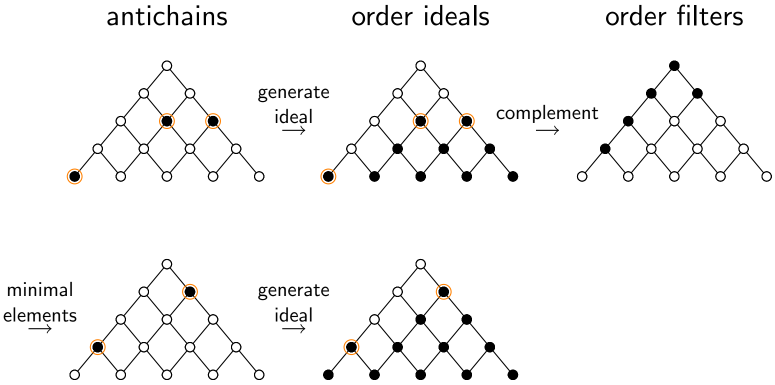

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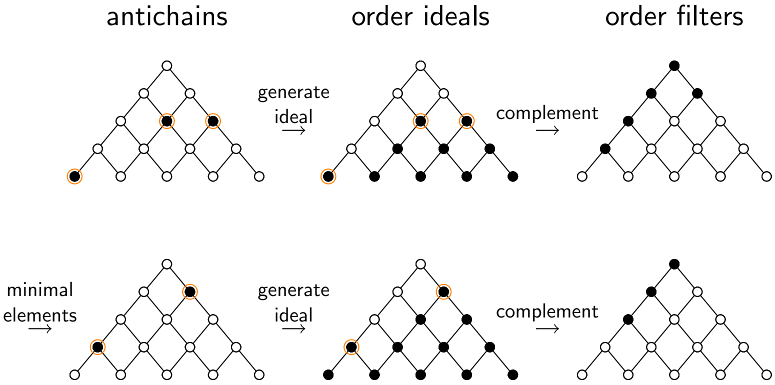

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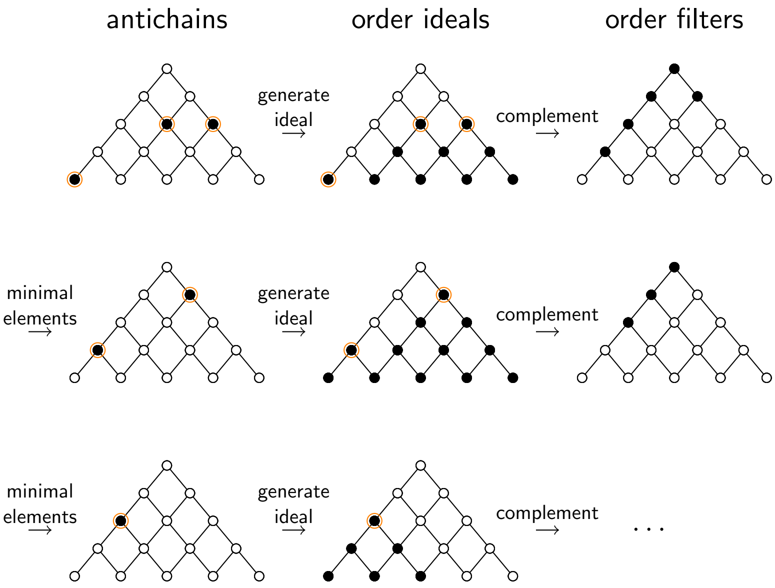

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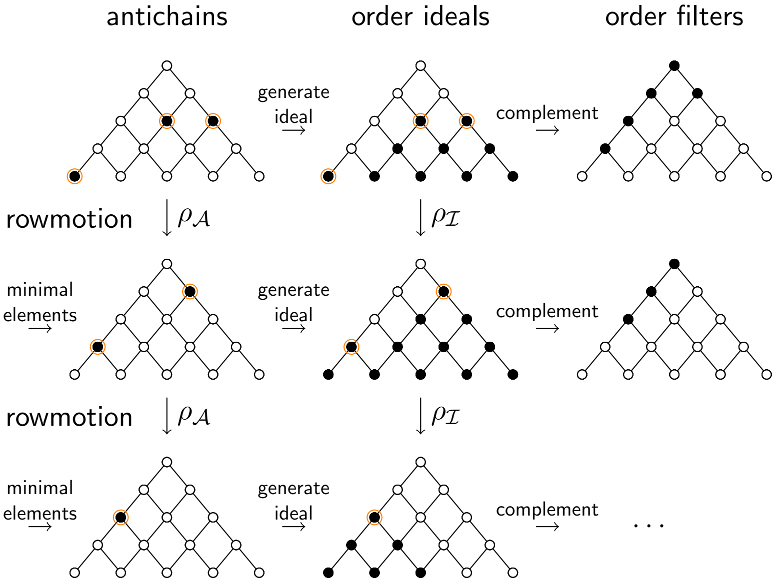
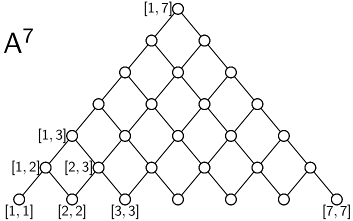


Type A root poset and Dyck paths

Let A^{n-1} denote the positive root poset of type A_{n-1}; equivalently, the set of intervals $\{[i,j] : 1 \le i \le j \le n-1\}$ ordered by inclusion.



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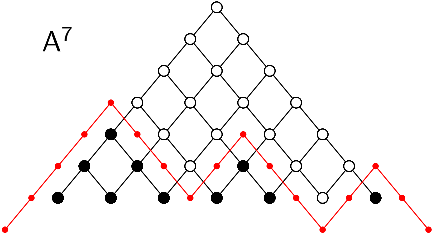

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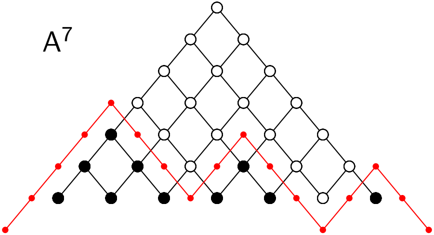

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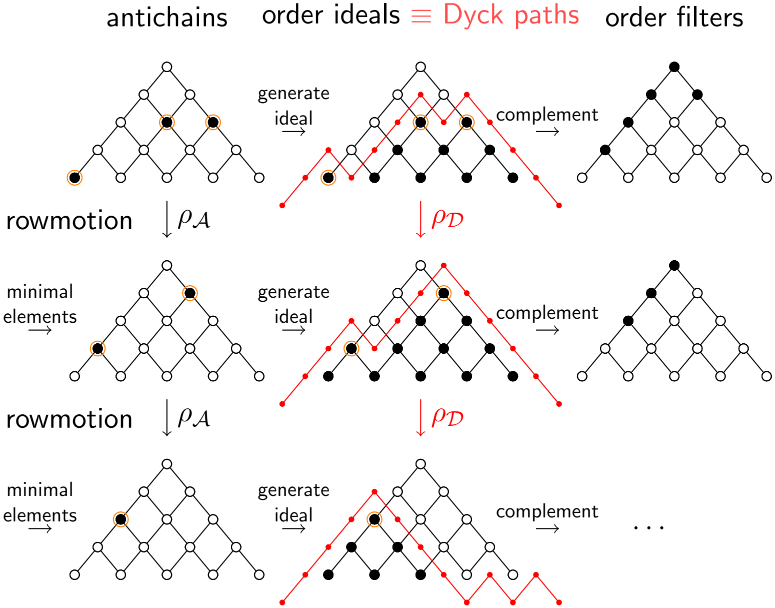
We can view rowmotion on ideals of A^{n-1} as an operation on Dyck paths $\rho_{\mathcal{D}} : \mathcal{D}_n \to \mathcal{D}_n$.

Rowmotion on Dyck paths



321-avoiding permutations

A permutation $\pi = \pi(1)\pi(2)\dots\pi(n) \in \mathcal{S}_n$ is 321-*avoiding* if there do not exist $i < j < k$ such that $\pi(i) > \pi(j) > \pi(k)$.

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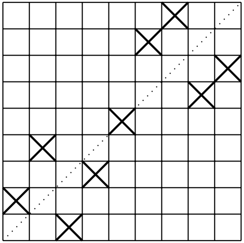
Example:

$\pi = 241358967$



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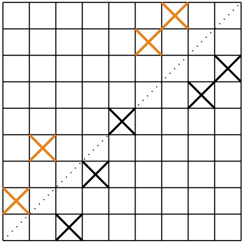

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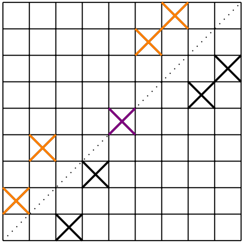

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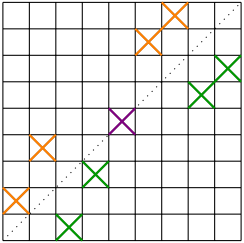

We say that $(i, \pi(i))$ is an excedance if $\pi(i) > i$, a fixed point if $\pi(i) = i$, and a deficiency if $\pi(i) < i$.

Properties of 321-avoiding permutations

Any $\pi \in \mathcal{S}_n(321)$ is uniquely determined by the positions and values of its excedances, which form an increasing subsequence.

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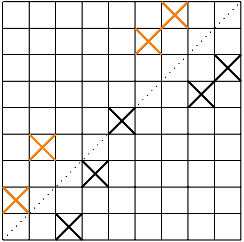

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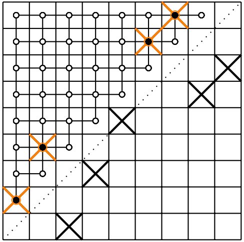

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$$\mathrm{Exc} : \mathcal{S}_n(321) \to \mathcal{A}(\mathsf{A}^{n-1}).$$

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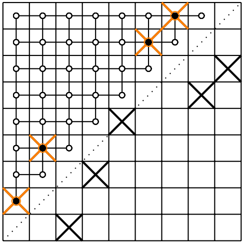
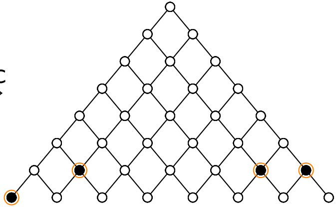


$\xrightarrow{\mathrm{Exc}}$

$\mathcal{A}(\mathsf{A}^{n-1})$

$= \text{antichains of } \mathsf{A}^{n-1}$



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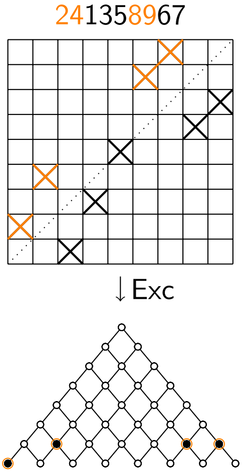

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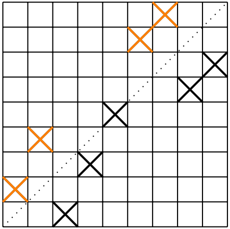
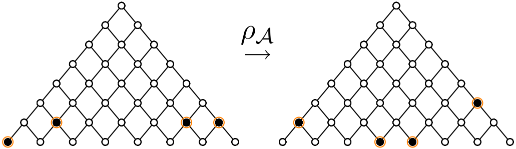


$\downarrow$ Exc



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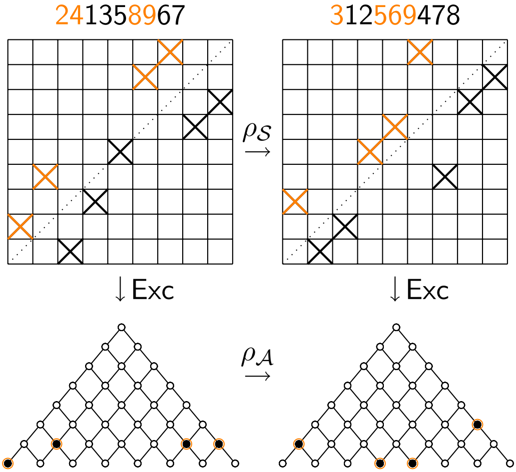

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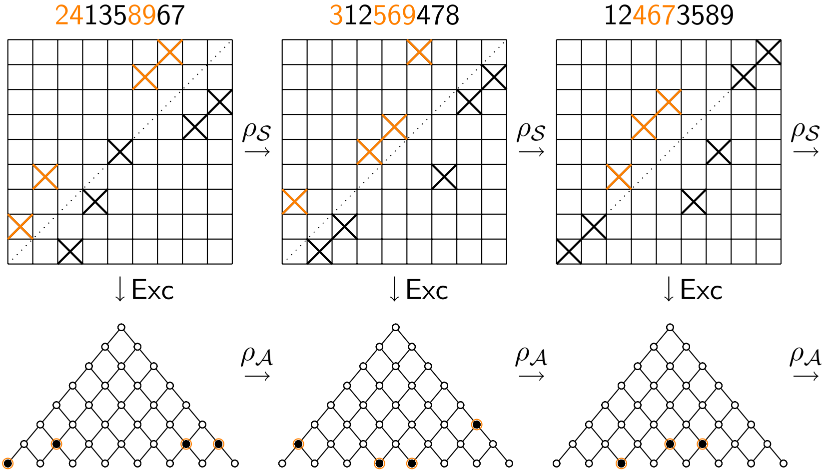

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If we consider the antichains of A^{n-1} given by the deficiencies of π instead, $\mathrm{Def}(\pi) := \mathrm{Exc}(\pi^{-1})$, then $\rho_{\mathcal{S}}$ is equivalent to *inverse rowmotion* of these antichains:

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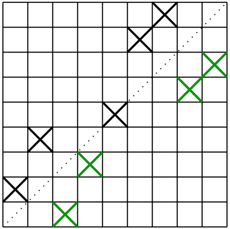

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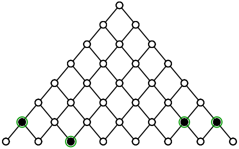

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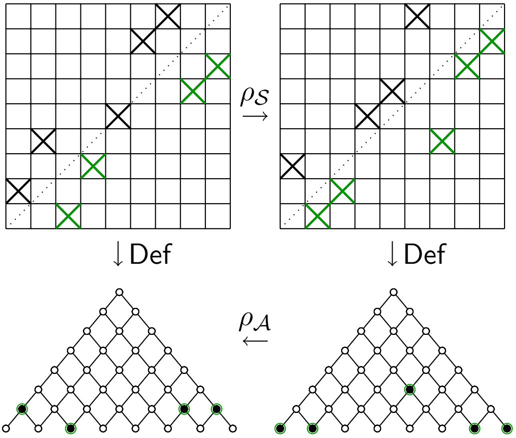

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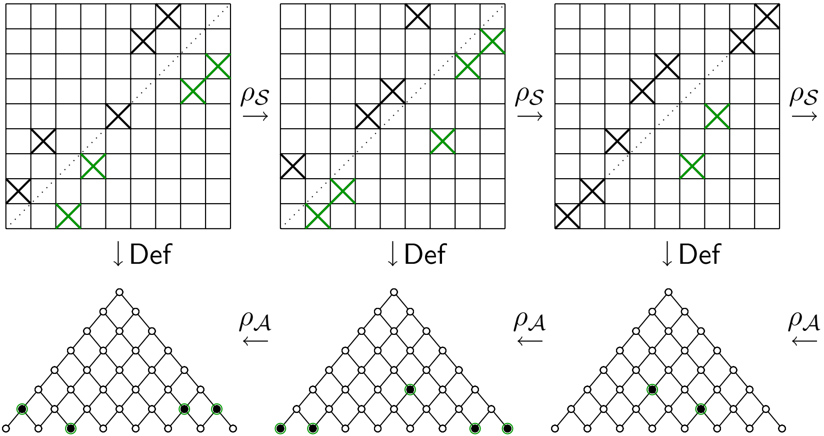
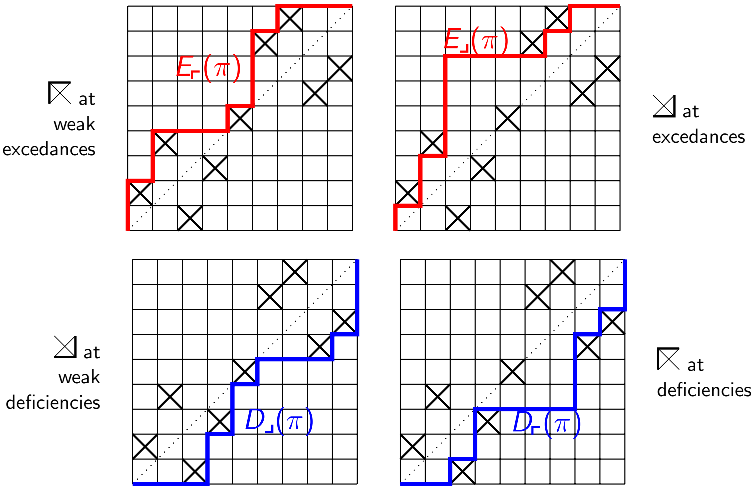


321-avoiding permutations and Dyck paths

Here are some bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ (Billey–Jockush–Stanley'93, Krattenthaler'01, E.'02):



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The rowmotion operation $\rho_{\mathcal{S}} : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ can be equivalently described as

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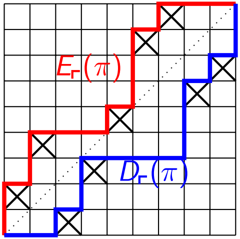

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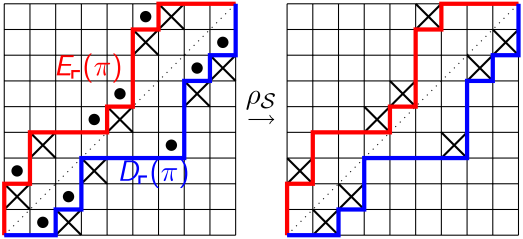

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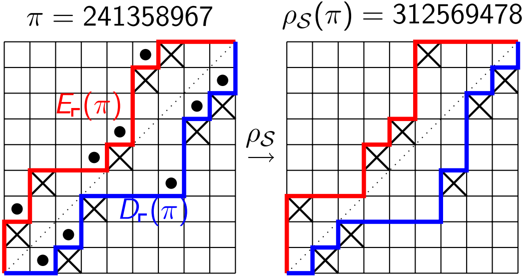

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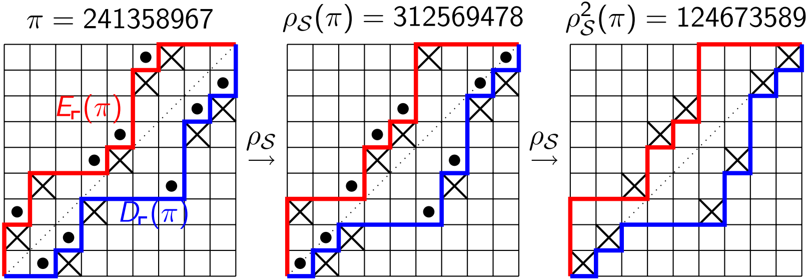

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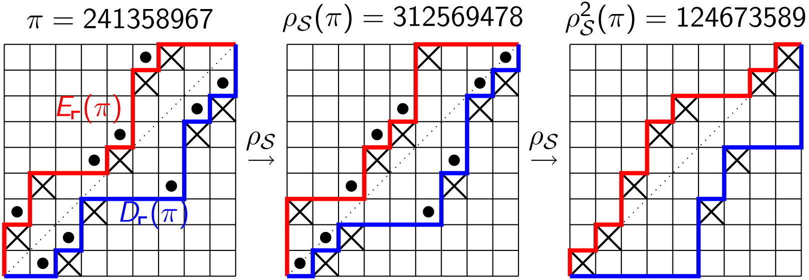

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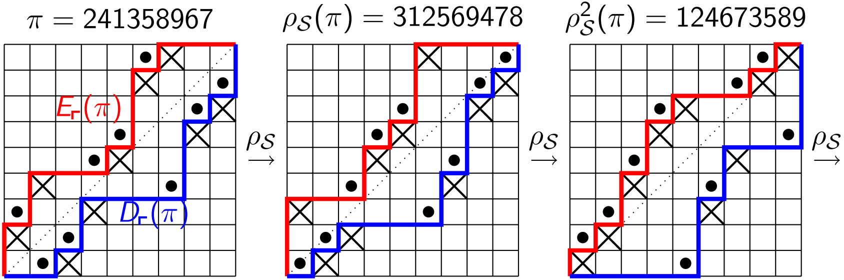

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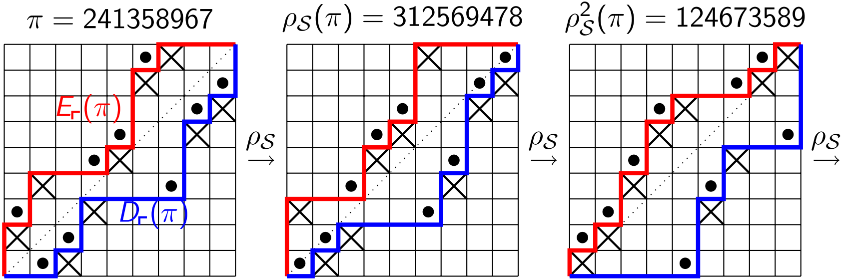

The map that sends $E_{\ulcorner}(\pi)$ to $D_{\ulcorner}(\pi)$ is called the *Lalanne–Kreweras* involution.

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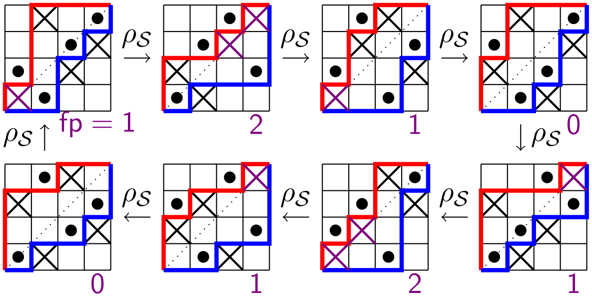
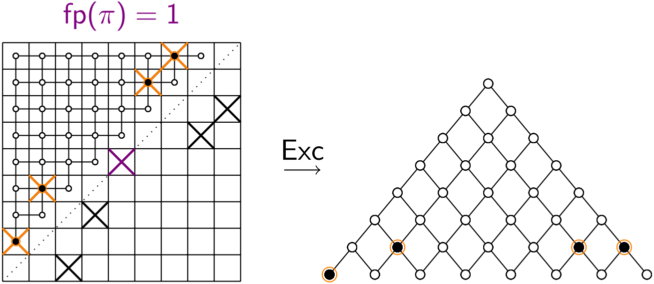


$$\frac{1+2+1+0+1+2+1+0}{8} = 1$$

Homomesy of fixed points

Note that the statistic fp does not correspond to a natural statistic on antichains.



The statistics h_i

Hopkins and Joseph define a family of statistics on antichains A of A^{n-1}:

$$h_i(A) = \sum_{j=1}^{i} \mathbb{1}_{[j,i]} + \sum_{j=i}^{n-1} \mathbb{1}_{[i,j]}, \quad \text{where } \mathbb{1}_x = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

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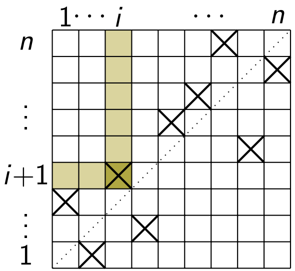
In terms of the permutation $\pi \in \mathcal{S}_n(321)$ such that $A = \operatorname{Exc}(\pi)$, this statistic is the number of crosses in the shaded region:



$h_3(314267958) = 2$

where the darker square in the corner is counted twice.

The statistics h_i

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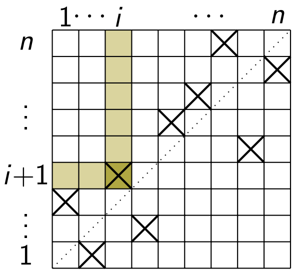
$$h_i(A) = \sum_{j=1}^{i} \mathbb{1}_{[j,i]} + \sum_{j=i}^{n-1} \mathbb{1}_{[i,j]}, \quad \text{where } \mathbb{1}_x = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In terms of the permutation $\pi \in \mathcal{S}_n(321)$ such that $A = \mathrm{Exc}(\pi)$, this statistic is the number of crosses in the shaded region:



$h_3(314267958) = 2$

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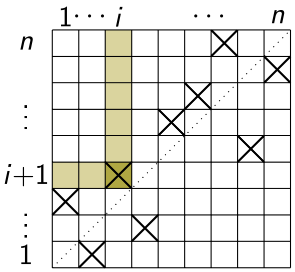

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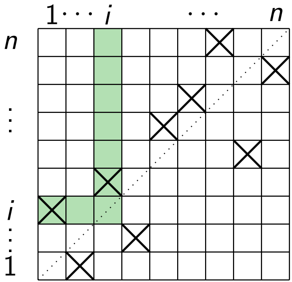

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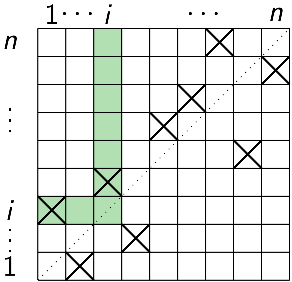

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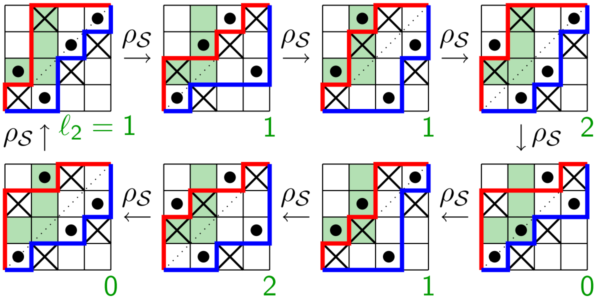

$$\frac{1+1+1+2+0+1+2+0}{8} = 1$$

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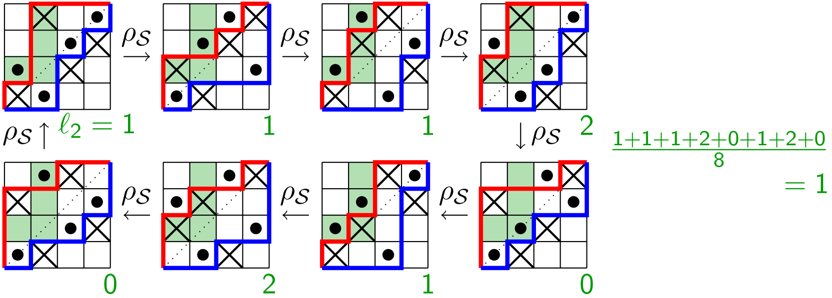

Using that h_i and ℓ_i are 1-mesic, we get another proof that fp is 1-mesic as well, since

$$\mathrm{fp}(\pi) = \sum_{i=1}^{n} \ell_i(\pi) - \sum_{i=1}^{n-1} h_i(\pi).$$

The sign statistic

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$$\mathrm{sgn}(\rho_{\mathcal{S}}(\pi)) = \begin{cases} \mathrm{sgn}(\pi) & \text{if } n \text{ is odd,} \\ -\mathrm{sgn}(\pi) & \text{if } n \text{ is even.} \end{cases}$$

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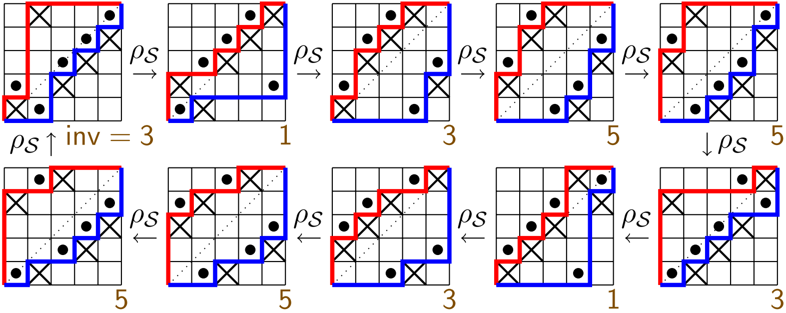

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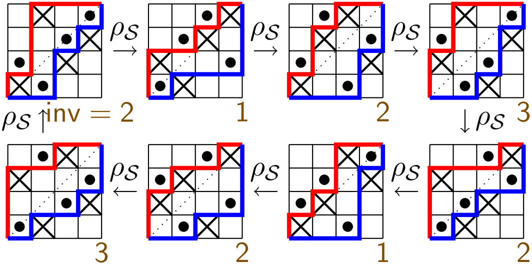

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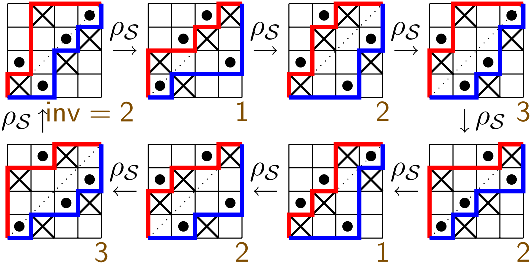

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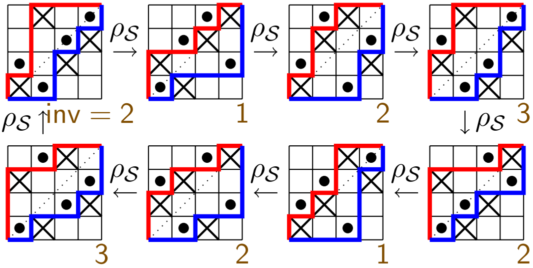

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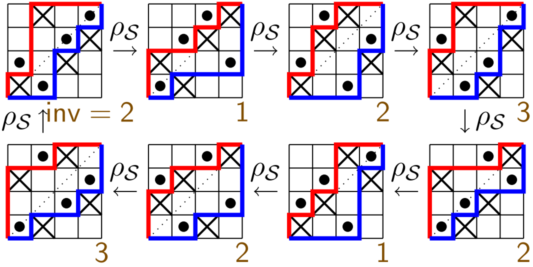

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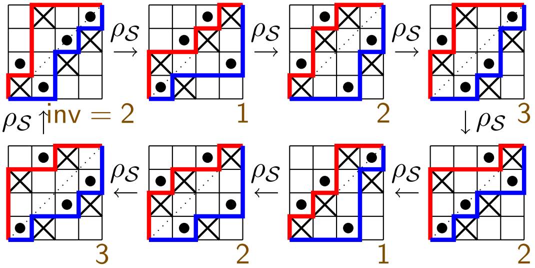

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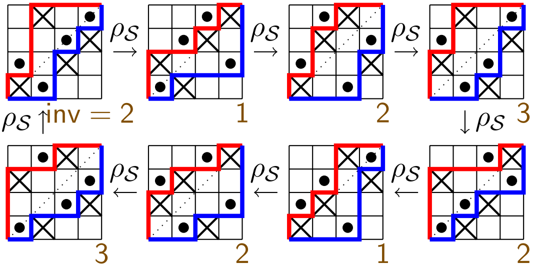

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And the map $\pi \mapsto \rho_{\mathcal{S}}(\pi^{-1})$ gives a sign-reversing involution.

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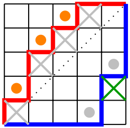

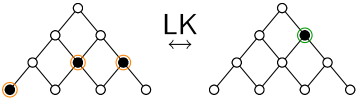

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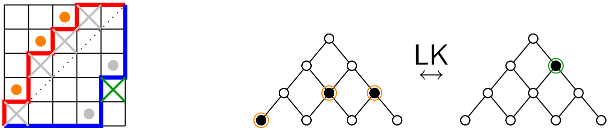

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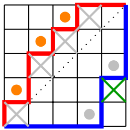

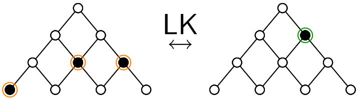

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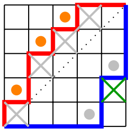

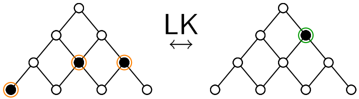

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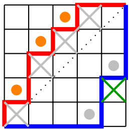

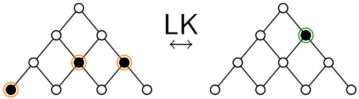

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by a classical result of Simion–Schmidt '85.

Promotion

Recall Schützenberger's promotion on standard Young tableaux:

$$T = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&7&8\\\hline 3&5&6&9&10\\\hline\end{array} \overset{\text{delete } 10}{\longrightarrow} \begin{array}{|c|c|c|c|c|}\hline 1&2&4&7&8\\\hline 3&5&6&9&\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1&2&4&7&8\\\hline 3&5&6&&9\\\hline\end{array}$$

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$\xrightarrow{\text{delete } 10}$

1	2	4	7	8
3	5	6	9	

$\rightarrow$

1	2	4	7	8
3	5	6		9

$\rightarrow$

1	2	4		8
3	5	6	7	9

$\rightarrow$

1	2		4	8
3	5	6	7	9

$\rightarrow$

1		2	4	8
3	5	6	7	9

$\rightarrow$

	1	2	4	8
3	5	6	7	9

$\xrightarrow{\text{place } 0}$

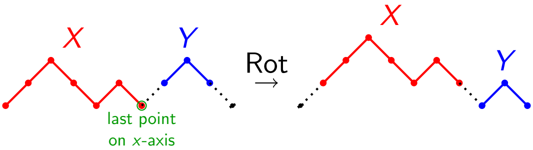
0	1	2	4	8
3	5	6	7	9

$\xrightarrow{+1}$

1	2	3	5	9
4	6	7	8	10

$= \text{Pro}(T)$

Define a rotation operation on Dyck paths:



Promotion and rotation

Via the standard bijections, promotion translates to rotation on Dyck paths and on non-crossing matchings:

1	2	4	7	8
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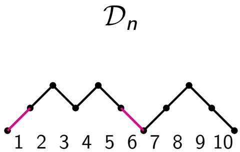
$\longrightarrow$



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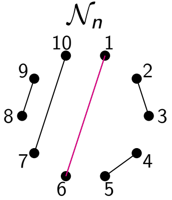

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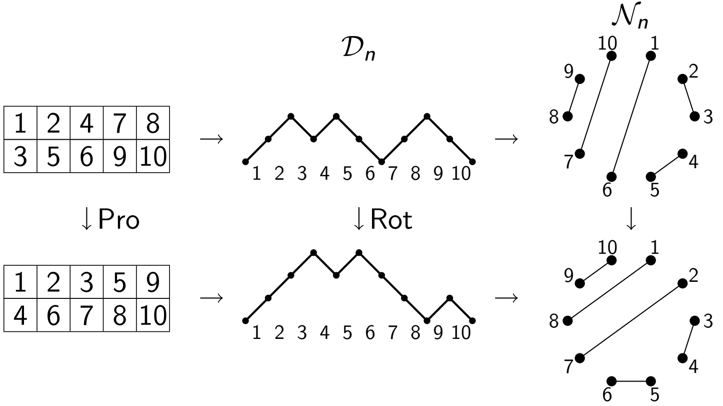

The Armstrong–Stump–Thomas bijection

Theorem (Armstrong–Stump–Thomas '13)

There is an equivariant bijection AST *between* $\mathcal{A}(\mathsf{A}^{n-1})$ *under rowmotion, and* $\mathcal{N}_n$ *(equivalently,* $\mathcal{D}_n$*) under rotation.*

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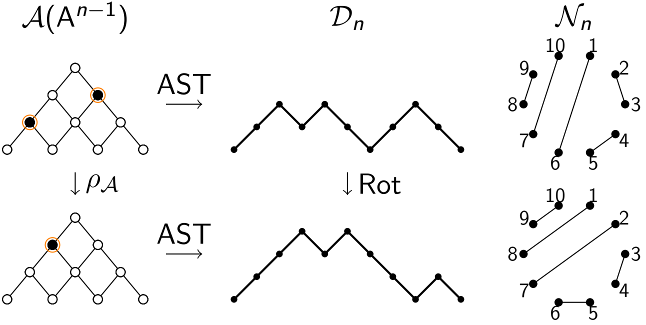

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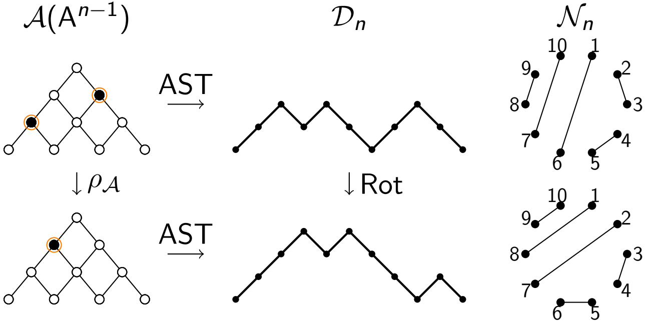
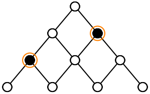


The bijection AST has a complicated description, and it is defined uniformly for all root systems.

A simpler description of AST

We can use 321-avoiding permutations to give a simple description of the AST bijection in type A:



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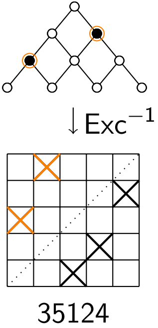

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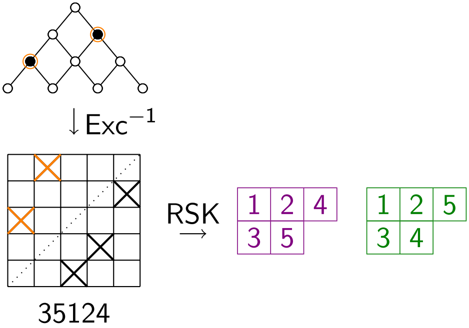

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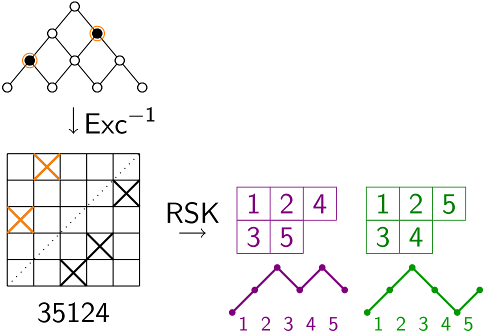

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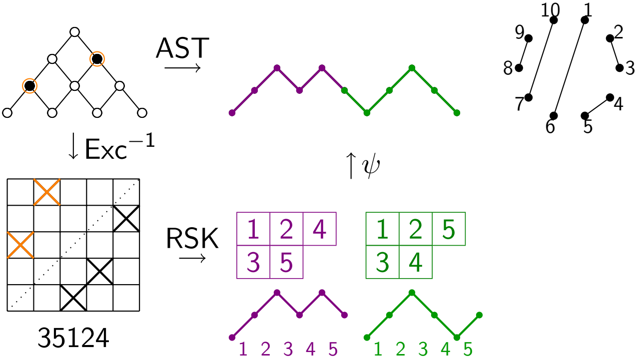

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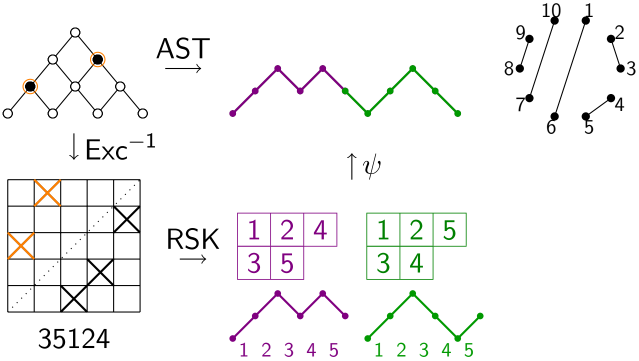

Theorem

$$\mathrm{AST} = \psi \circ \mathrm{RSK} \circ \mathrm{Exc}^{-1}$$

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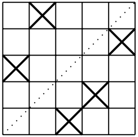


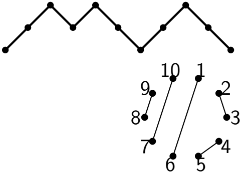
35124

RSK →

1	2	4
3	5	

1	2	5
3	4	

$\xrightarrow{\psi}$



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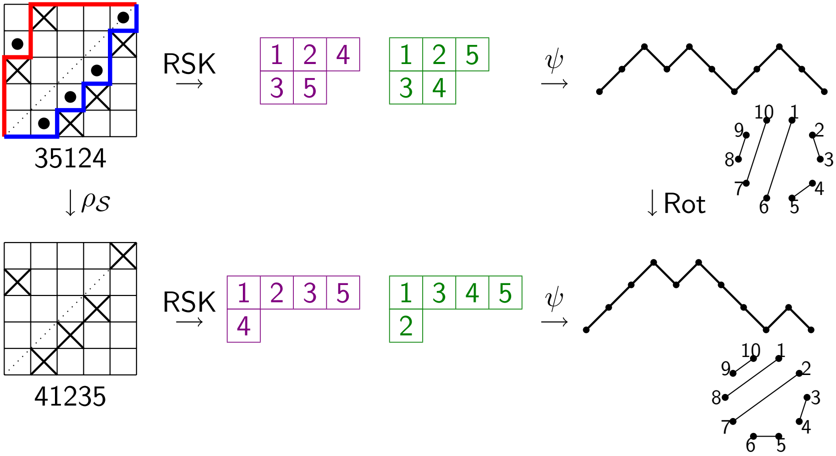

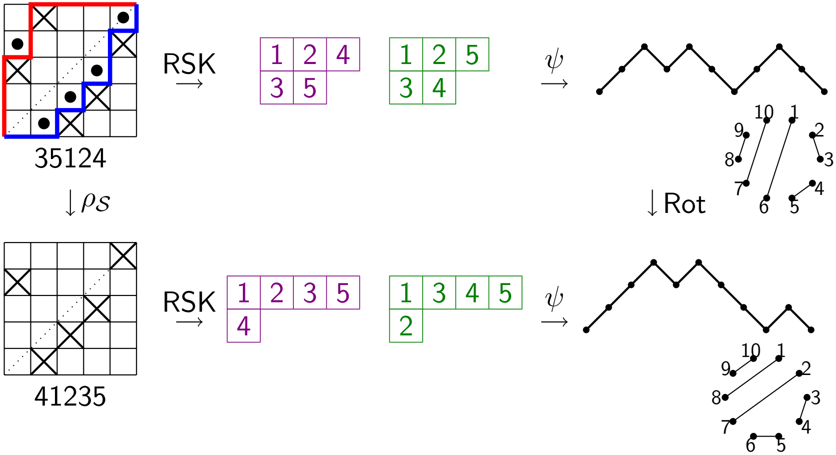
35124
RSK
1 2 4
3 5
1 2 5
3 4
ψ
10 1 9 2 8 3 7 4 6 5
↓ ρS
↓ Rot
41235
RSK
1 2 3 5
4
1 3 4 5
2
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THANK YOU!