LIST OF PUBLICATIONS WITH ABSTRACTS
in reverse chronological order

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Papers available at `http://www.math.dartmouth.edu/~sergi`

1. S.E., **On individual leaf depths of trees**, *Discrete Appl. Math.* 353 (2024), 151–180.

   We explore a generating function trick which allows us to keep track of infinitely many statistics using finitely many variables, by recording their individual distributions rather than their joint distributions. Building on previous work of Panholzer and Prodinger, we apply this method to study the depth distributions of individual nodes or leaves in rooted binary trees, plane trees, noncrossing trees, and increasing trees; the height distributions of individual vertices and individual steps in Dyck paths; and the number of diagonals separating two fixed sides of a convex polygon in a triangulation or dissection. We obtain both exact and asymptotic results, which sometimes refine known formulas or provide combinatorial proofs of results from the probability literature.

2. S.E., **Descents on nonnesting multipermutations**, *European J. Combin.* (2023), `https://doi.org/10.1016/j.ejc.2023.103846`.

   Motivated by recent results on quasi-Stirling permutations, which are permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ that avoid the "crossing" patterns 1212 and 2121, we consider nonnesting permutations, defined as those that avoid the patterns 1221 and 2112 instead. We show that the polynomial giving the distribution of the number of descents on nonnesting permutations is a product of an Eulerian polynomial and a Narayana polynomial. It follows that, rather unexpectedly, this polynomial is palindromic. We provide bijective proofs of these facts by composing various transformations on Dyck paths, including the Lalanne–Kreweras involution.

3. S.E., **The Distribution of Descents on Nonnesting Permutations**, *Proceedings of FPSAC 2023, Sém. Lothar. Combin.* 89B.2 (2023), 12 pp.

4. Ben Adenbaum and S.E., **Rowmotion on 321-avoiding permutations**, to appear in *Electron. J. Combin.*

   We give a natural definition of rowmotion for 321-avoiding permutations, by translating, through bijections involving Dyck paths and the Lalanne–Kreweras involution, the analogous notion for antichains of the positive root poset of type A. We prove that some permutation statistics, such as the number of fixed points, are homomesic under rowmotion, meaning that they have a constant average over its orbits.

   Our setting also provides a more natural description of the celebrated Armstrong–Stump–Thomas equivariant bijection between antichains and non-crossing matchings in types A and B, by showing that it is equivalent to the Robinson–Schensted–Knuth correspondence on 321-avoiding permutations permutations.

5. Ben Adenbaum and S.E., **Rowmotion on 321-Avoiding Permutations**, *Sém. Lothar. Combin.* 89B.16 (2023), 12 pp.

6. S.E., M. Plante, T. Roby and B. Sagan, **Rowmotion on fences**, *Algebr. Comb.* 6 (2023), 17–36.

   A fence is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$ and covers $x_1 \lhd x_2 \lhd \ldots \lhd x_a \rhd x_{a+1} \rhd \ldots \rhd x_b \lhd x_{b+1} \lhd \cdots$, where $a, b, \ldots$ are positive integers. We investigate rowmotion on antichains and ideals of F. In particular, we show that orbits of antichains can be visualized using tilings. This permits us to prove various homomesy results for the number of elements of an antichain or ideal in an orbit. Rowmotion on fences also exhibits a new phenomenon, which we call homometry, where the value of a statistic is constant on orbits of the same size. Along the way, we prove a general homomesy result for all self-dual posets. We end with some conjectures and avenues for future research.

7. S.E., M. Plante, T. Roby and B. Sagan, **Rowmotion on fences**, *Proceedings of FPSAC 2022, Sém. Lothar. Combin.* 86B.40 (2022), 12 pp.

8. S. Corteel, S.E. and C. Savage, **Partitions with constrained ranks and lattice paths**, *ECA* 3:3 (2023) Article #S2R18.

   In this paper we study partitions whose successive ranks belong to a given set. We enumerate such partitions while keeping track of the number of parts, the largest part, the side of the Durfee square, and the height of the Durfee rectangle. We also obtain a new bijective proof of a result of Andrews and Bressoud that the number of partitions of N with all ranks at least $1-\ell$ equals the number of partitions of N with no parts equal to $\ell+1$, for $\ell \geq 0$, which allows us to refine it by the above statistics. Combining Foata's second fundamental transformation for words with Greene and Kleitman's mapping for subsets, interpreted in terms of lattice paths, we obtain enumeration formulas for partitions whose successive ranks satisfy certain constraints, such as being bounded by a constant. While some of these formulas have alternative proofs using results of Krattenthaler–Mohanty and Burge, our approach seems to be new.

9. S.E., **On a dart game of Niedermaier**, *Adv. in Appl. Math.* 145 (2023) 102483.

   We analyze a game introduced by Andy Niedermaier, where m players take turns throwing a dart at a dartboard. A player is eliminated unless his dart lands closer to the center than all previously thrown darts, in which case he goes to the back of the line, until only one player remains. Using generating functions, we determine the distribution of the number of throws in the game, and we obtain a recursive formula to compute the probability that each player wins.

10. S.E., **Walks in Simplices, Cylindric Tableaux, and Asymmetric Exclusion Processes**, *Proceedings of FPSAC 2022, Sém. Lothar. Combin.* 86B.8 (2022), 12 pp.

    We establish bijections between three classes of combinatorial objects that have been studied in different contexts: lattice walks in simplicial regions as introduced by Mortimer–Prellberg, standard cylindric tableaux as introduced by Gessel–Krattenthaler and Postnikov, and sequences of states in the totally asymmetric simple exclusion process. This perspective gives new insights into these objects, providing a vehicle to translate enumerative results from lattice walks to tableaux, and to interpret symmetries that are natural in one setting (e.g. conjugation of tableaux) as involutions in another. Specifically, it allows us to use a cylindric analogue of the Robinson–Schensted correspondence to give an alternative bijective proof of a recent result of Courtiel, Elvey Price and Marcovici relating forward and backward walks in simplices.

11. S.E., **Counting lattice paths by crossings and major index II: tracking descents via two-rowed arrays**, *Sém. Lothar. Combin.* 87B (2022), Art. #2, 33 pp.

    We present refined enumeration formulas for lattice paths in $\mathbb{Z}^2$ with two kinds of steps, by keeping track of the number of descents (i.e., turns in a given direction), the major index (i.e., the sum of the positions of the descents), and the number of crossings. One formula considers crossings between a path and a fixed line; the other considers crossings between two paths. Building on the first paper of the series, which used lattice path bijections to give the enumeration with respect to major index and crossings, we obtain a refinement that keeps track of the number of descents. The proof is based on new bijections which rely on certain two-rowed arrays that were introduced by Krattenthaler.

12. S.E. and B. Sagan, **Partial Rank Symmetry of Distributive Lattices for Fences**, *Ann. Comb.* 27 (2023), 433–454.

    Associated with any composition $\beta = (a, b, \ldots)$ is a corresponding fence poset $F(\beta)$ whose covering relations are $x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots$. The distributive lattice $L(\beta)$ of all lower order ideals of $F(\beta)$ is important in the theory of cluster algebras. In addition, its rank generating function $r(q; \beta)$ is used to define q-analogues of rational numbers. Kantarcı Oğuz and Ravichandran recently showed that its coefficients satisfy an interlacing condition, proving a conjecture of McConville, Smyth and Sagan, which in turn implies a previous conjecture of Morier-Genoud and Ovsienko that $r(q; \beta)$ is unimodal. We show that, when β has an odd number of parts, then the polynomial is also partially symmetric: the number of ideals of $F(\beta)$ of size k equals the number of filters of size k, when k is below a certain value. Our proof is completely bijective. Kantarcı Oğuz and Ravichandran also introduced a circular version of fences and proved, using algebraic techniques, that the distributive lattice for such a poset is rank symmetric. We give a bijective proof of this result as well. We end with some questions and conjectures raised by this work.

13. S.E., **Counting lattice paths by crossings and major index I: the corner-flipping bijections**, *Combinatorial Theory* 2 (2022), #14.

    We solve two problems regarding the enumeration of lattice paths in $\mathbb{Z}^2$ with steps $(1, 1)$ and $(1, -1)$ with respect to the major index, defined as the sum of the positions of the valleys, and to the number of certain crossings. The first problem considers crossings of a single path with a fixed horizontal line. The second one counts pairs of paths with respect to the number of times they cross each other. Our proofs introduce lattice path bijections with convenient visual descriptions, and the answers are given by remarkably simple formulas involving q-binomial coefficients.

14. R. Domagalski, S.E., J. Liang, Q. Minnich, B. Sagan, J. Schmidt and A. Sietsema, **Cyclic pattern containment and avoidance**, *Adv. in Appl. Math.* 135 (2022), 102320.

    The study of pattern containment and avoidance for linear permutations is a well-established area of enumerative combinatorics. A cyclic permutation is the set of all rotations of a linear permutation. Vella and Callan independently initiated the study of permutation avoidance in cyclic permutations and characterized the avoidance classes for all single permutations of length 4. We continue this work. In particular, we derive results about avoidance of multiple patterns of length 4, and we determine generating functions for the cyclic descent statistic on these classes. We also consider consecutive pattern containment, and relate the generating functions for the number of occurrences of certain linear and cyclic patterns. Finally, we end with various open questions and avenues for future research.

15. S.E. and E. Deutsch, **The degree of asymmetry of sequences**, *ECA* 2:1 (2022), Article S2R7.

    We explore the notion of degree of asymmetry for integer sequences and related combinatorial objects. The degree of asymmetry is a new combinatorial statistic that measures how far an object is from being symmetric. We define this notion for compositions, words, matchings, binary trees and permutations, we find generating functions enumerating these objects with respect to their degree of asymmetry, and we describe the limiting distribution of this statistic in each case.

16. S.E., R. Flórez and J. L. Ramírez, **Enumerating symmetric peaks in non-decreasing Dyck paths**, *Ars Math. Contemp.* 21 (2021) #P2.04.

    Local maxima and minima of a Dyck path are called peaks and valleys, respectively. A Dyck path is non-decreasing if the heights (y-coordinates) of its valleys increase from left to right. A peak is symmetric if it is surrounded by two valleys (or endpoints of the path) at the same height. In this paper we give multivariate generating functions, recurrence relations, and closed formulas to count the number of symmetric and asymmetric peaks in non-decreasing Dyck paths. Finally, we use Riordan arrays to study weakly symmetric peaks, namely those for which the valley preceding the peak is at least as high as the valley following it.

17. S.E., **Bijections for restricted inversion sequences and permutations with fixed points**, *Australas. J. Combin.* 80 (2021), 106–115.

    We provide a bijective proof of a formula of Auli and the author (DMTCS 2019) expressing the number of inversion sequences with no three consecutive equal entries in terms of the number of non-derangements, that is, permutations with fixed points. Additionally, we give bijective proofs of two simple recurrences for the number of non-derangements.

18. S.E., **Descents on quasi-Stirling permutations**, *J. Combin. Theory Ser. A* 180 (2021) 105429.

    Stirling permutations were introduced by Gessel and Stanley in 1978, who enumerated them by the number of descents to give a combinatorial interpretation of certain polynomials related to Stirling numbers. A natural extension of these permutations are quasi-Stirling permutations, which can be viewed as labeled noncrossing matchings. They were recently introduced by Archer et al., motivated by the fact that the Koganov–Janson correspondence between Stirling permutations and labeled increasing plane trees extends to a bijection between quasi-Stirling permutations and the same set of trees without the increasing restriction.

    In this paper we prove a conjecture of Archer et al. stating that there are $(n+1)^{n-1}$ quasi-Stirling permutations of size n having n descents. More generally, we give the generating function for quasi-Stirling permutations by the number of descents, expressed as a compositional inverse of the generating function of Eulerian polynomials. We also find the analogue for quasi-Stirling permutations of the main result from Gessel and Stanley's paper. We prove that the distribution of descents on these permutations is asymptotically normal, and that the roots of the corresponding quasi-Stirling polynomials are all real, in analogy to Bóna's results for Stirling permutations.

    Finally, we generalize our results to a one-parameter family of permutations that extends k-Stirling permutations, and we refine them by also keeping track of the number of ascents and the number of plateaus.

19. S.E., **Enumerating Descents on Quasi-Stirling Permutations and Plane Trees**, *Extended Abstracts EuroComb 2021*, Trends in Mathematics, vol 14. Birkhäuser, Cham., pp 32-37, `https://doi.org/10.1007/978-3-030-83823-2_6`.

20. S.E., **Symmetric peaks and symmetric valleys in Dyck paths**, *Discrete Math.* 344 (2021) 112364.

    The notion of symmetric and asymmetric peaks in Dyck paths was introduced by Flórez and Rodríguez, who counted the total number of such peaks over all Dyck paths of a given length. In this paper we generalize their results by giving multivariate generating functions that keep track of the number of symmetric peaks and the number of asymmetric peaks, as well as the widths of these peaks. We recover a formula of Denise and Simion as a special case of our results.

    We also consider the analogous but more intricate notion of symmetric valleys. We find a continued fraction expression for the generating function of Dyck paths with respect to the number of symmetric valleys and the sum of their widths, which provides an unexpected connection between symmetric valleys and statistics on ordered rooted trees. Finally, we enumerate Dyck paths whose peak or valley heights satisfy certain monotonicity and unimodality conditions, using a common framework to recover some known results, and relating our questions to the enumeration of certain classes of column-convex polyominoes.

21. T. Watkins, E. Lim, M. Petkovic, S.E., ..., R. Schwarz, N. McGranahan and C. Swanton, **Pervasive chromosomal instability and karyotype order during tumour evolution**, *Nature* 587, 126–132 (2020).

    Cancer chromosomal instability (CIN) results from dynamic changes to chromosome number and structure. The resulting diversity in somatic copy number alterations (SCNA) may provide the variation necessary for cancer evolution. Multi-sample phasing and SCNA analysis of 1421 samples from 394 tumours across 24 cancer types revealed ongoing CIN resulting in pervasive SCNA heterogeneity. Parallel evolutionary events, causing disruption to the same genes, such as *BCL9*, *ARNT/HIF1B*, *TERT* and *MYC*, within separate subclones were present in 35% of tumours. Most recurrent losses occurred prior to whole genome doubling (WGD), a clonal event in 48% of tumours. However, loss of heterozygosity at the human leukocyte antigen locus and loss of 8p to a single haploid copy recurred at significant subclonal frequencies, even in WGD tumours, likely reflecting ongoing karyotype remodeling. Focal amplifications affecting 1q21 (*BCL9, ARNT*), 5p15.33 (*TERT*), 11q13.3 (*CCND1*), 19q12 (*CCNE1*) and 8q24.1 (*MYC*) were frequently subclonal and exhibited an illusion of clonality within single samples. Analysis of an independent series of 1024 metastatic samples revealed enrichment for 14 focal SCNAs in metastatic samples, including late gains of 8q24.1 (*MYC*) in clear cell renal carcinoma and 11q13.3 (*CCND*1) in HER2-positive breast cancer. CIN may enable ongoing selection of SCNAs, manifested as ordered events, often occurring in parallel, throughout tumour evolution.

22. J.S. Auli and S.E., **Wilf equivalences between vincular patterns in inversion sequences**, *Appl. Math. Comput.* 388 (2020) 125514.

    Inversion sequences are finite sequences of non-negative integers, where the value of each entry is bounded from above by its position. They provide a useful encoding of permutations. Patterns in inversion sequences have been studied by Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck in the classical case, where patterns can occur in any positions, and by Auli–Elizalde in the consecutive case, where only adjacent entries can form an occurrence of a pattern. These papers classify classical and consecutive patterns of length 3 into Wilf equivalence classes according to the number of inversion sequences avoiding them.

    In this paper we consider vincular patterns in inversion sequences, which, in analogy to Babson–Steingrímsson patterns in permutations, require only certain entries of an occurrence to be adjacent, and thus generalize both classical and consecutive patterns. Solving three conjectures of Lin and Yan, we provide a complete classification of vincular patterns of length 3 in inversion sequences into Wilf equivalence classes, and into more restrictive classes that consider the number of occurrences of the pattern and the positions of such occurrences. We find the first known instance of patterns in inversion sequences where these two more restrictive classes do not coincide.

23. S.E., **A simple bijective proof of a familiar derangement recurrence**, *Fibonacci Quart.* 59 (2021), 150–151.

    It is well known that the derangement numbers d_n, which count permutations of length n with no fixed points, satisfy the recurrence $d_n = nd_{n-1} + (-1)^n$ for $n \geq 1$. Combinatorial proofs of this formula have been given by Remmel, Wilf, Désarménien and Benjamin–Ornstein. Here we present yet another, arguably simpler bijective proof.

24. S.E., **The degree of symmetry of lattice paths**, *Ann. Comb.* 25 (2021), 877–911.

    The degree of symmetry of a combinatorial object, such as a lattice path, is a measure of how symmetric the object is. It typically ranges from zero, if the object is completely asymmetric, to its size, if it is completely symmetric. We study the behavior of this statistic on Dyck paths and grand Dyck paths, with symmetry described by reflection along a vertical line through their midpoint; partitions, with symmetry given by conjugation; and certain compositions interpreted as bargraphs. We find expressions for the generating functions for these objects with respect to their degree of symmetry, and their semilength or semiperimeter, deducing in most cases that, asymptotically, the degree of symmetry has a Rayleigh or half-normal limiting distribution. The resulting generating functions are often algebraic, with the notable exception of Dyck paths, for which we conjecture that it is D-finite (but not algebraic), based on a functional equation that we obtain using bijections to walks in the plane.

25. S.E., **Measuring symmetry in lattice paths and partitions**, *Proceedings of FPSAC 2020, Sém. Lothar. Combin.* 84B (2020), Article #26, 12 pp.

26. J. Bloom, S.E. and Y. Roichman, **On cyclic Schur-positive sets of permutations**, *Electron. J. Combin.* 27(2) (2020), #P2.6.

    We introduce a notion of *cyclic Schur-positivity* for sets of permutations, which naturally extends the classical notion of Schur-positivity, and it involves the existence of a bijection from permutations to standard Young tableaux that preserves the cyclic descent set. Cyclic Schur-positive sets of permutations are always Schur-positive, but the converse does not hold, as exemplified by inverse descent classes, Knuth classes and conjugacy classes.

    In this paper we show that certain classes of permutations invariant under either horizontal or vertical rotation are cyclic Schur-positive. The proof unveils a new equidistribution phenomenon of descent sets on permutations, provides affirmative solutions to conjectures by the last two authors and by Adin–Gessel–Reiner–Roichman, and yields new examples of Schur-positive sets.

27. J.S. Auli and S.E., **Consecutive patterns in inversion sequences II: avoiding patterns of relations**, *J. Integer Seq.* 22 (2019), Art. 19.7.5.

    Inversion sequences are integer sequences $e = e_1e_2 \dots e_n$ such that $0 \le e_i < i$ for each i. The study of patterns in inversion sequences was initiated by Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck in the classical (non-consecutive) case, and later by Auli–Elizalde in the consecutive case, where the entries of a pattern are required to occur in adjacent positions. In this paper we continue this investigation by considering *consecutive patterns of relations*, in analogy to the work of Martinez–Savage in the classical case. Specifically, given two binary relations $R_1, R_2 \in \{\le, \ge, <, >, =, \ne\}$, we study inversion sequences e with no subindex i such that $e_iR_1e_{i+1}R_2e_{i+2}$. By enumerating such inversion sequences according to their length, we obtain well-known quantities such as Catalan numbers, Fibonacci numbers and central polynomial numbers, relating inversion sequences to other combinatorial structures. We also classify consecutive patterns of relations into Wilf equivalence classes, according to the number of inversion sequences avoiding them, and into more restrictive classes that consider the positions of the occurrences of the patterns.

    As a byproduct of our techniques, we obtain a simple bijective proof of a result of Baxter–Shattuck and Kasraoui about Wilf-equivalence of vincular patterns, and we prove a conjecture of Martinez and Savage, as well as related enumeration formulas for inversion sequences satisfying certain unimodality conditions.

28. J.S. Auli and S.E., **Consecutive patterns in inversion sequences**, *Discrete Math. Theor. Comput. Sci.* 21:2 (2019), #6, 22 pp.

    An inversion sequence of length n is an integer sequence $e = e_1e_2 \dots e_n$ such that $0 \le e_i < i$ for each i. Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck began the study of patterns in inversion sequences, focusing on the enumeration of those that avoid classical patterns of length 3. We initiate an analogous systematic study of *consecutive* patterns in inversion sequences, namely patterns whose entries are required to occur in adjacent positions. We enumerate inversion sequences that avoid consecutive patterns of length 3, and generalize some results to patterns of arbitrary length. Additionally, we study the notion of Wilf equivalence of consecutive patterns in inversion sequences, as well as generalizations of this notion analogous to those studied for permutation patterns. We classify patterns of length up to 4 according to the corresponding Wilf equivalence relations.

29. J.S. Auli and S.E., **Inversion sequences avoiding consecutive patterns**, *Proceedings of FPSAC 2020, Sém. Lothar. Combin.* 84B (2020), Article #27, 12 pp.

30. S.E. and K. Moore, **Characterizations and enumerations of patterns of signed shifts**, *Discrete Appl. Math* 277 (2020), 92–114.

    Shift maps can be interpreted as maps from the unit interval to itself sending x to the fractional part of Nx, whose graph consists of N segments of positive slope. Signed shifts generalize shift maps by allowing some of these segments to have negative slope. Permutations realized by the relative order of the elements in the orbits of these maps have been studied recently by Amigó, Archer and Elizalde. In this paper, we give a complete characterization of the permutations (also called patterns) realized by signed shifts. In the case of the negative shift, which is the signed shift having only negative slopes, we use our characterization to give an exact enumeration of these patterns. Finally, we improve the best known bounds for the number of patterns realized by the tent map, and calculate the topological entropy of signed shifts using these combinatorial methods.

31. S.E. and J. Troyka **Exact and asymptotic enumeration of cyclic permutations according to descent set**, *J. Combin. Theory Ser. A* 165 (2019), 360–391.

    Using a result of Gessel and Reutenauer, we find a simple formula for the number of cyclic permutations with a given descent set, by expressing it in terms of ordinary descent numbers (i.e., those counting all permutations with a given descent set). We then use this formula to show that, for almost all sets $I \subseteq [n-1]$, the fraction of size-n permutations with descent set I which are n-cycles is asymptotically $1/n$. As a special case, we recover a result of Stanley for alternating cycles. We also use our formula to count n-cycles with no two consecutive descents.

32. R. Adin, S.E., and Y. Roichman, **Cyclic descents for near-hook and two-row shapes**, *European J. Combin.* 79 (2019), 152–178.

    A notion of cyclic descents on standard Young tableaux (SYT) of rectangular shape was introduced by Rhoades, and extended to certain skew shapes by the last two authors. The cyclic descent set restricts to the usual descent set when the largest value is ignored, and has the property that the number of SYT of a given shape with a given cyclic descent set D is invariant under cyclic shifts of the entries of D. Following these results, the existence of cyclic descent sets for standard Young tableaux of any skew shape other than a ribbon was conjectured by the authors, and recently proved by Adin, Reiner and Roichman. Unfortunately, the proof does not provide a natural definition of the cyclic descent set for a specific tableau.

    In this paper we explicitly describe cyclic descent sets for SYT of (possibly skew) shapes which either have exactly two rows or are near-hooks, i.e., are one cell away from a hook. Our definition provides a constructive combinatorial proof of the existence of cyclic descent sets for these shapes, and coincides with that of Rhoades for two-row rectangular shapes. We also show that cyclic descent sets for near-hook shaped tableaux are unique.

33. **A Markov chain for numerical chromosomal instability in clonally expanding populations** (with A. Laughney and S. Bakhoum), *PLoS Comput. Biol.* 14(9) (2018): e1006447.

    Cancer cells frequently undergo chromosome missegregation events during mitosis, whereby the copies of a given chromosome are not distributed evenly among the two daughter cells, thus creating cells with heterogeneous karyotypes. A stochastic model tracing cellular karyotypes derived from clonal populations over hundreds of generations was recently developed and experimentally validated, and it was capable of predicting favorable karyotypes frequently observed in cancer. Here, we construct and study a Markov chain that precisely describes karyotypic evolution during clonally expanding cancer cell populations. The Markov chain allows us to directly predict the distribution of karyotypes and the expected size of the tumor after many cell divisions without resorting to computationally expensive simulations. We determine the limiting karyotype distribution of an evolving tumor population, and quantify its dependency on several key parameters including the initial karyotype of the founder cell, the rate of whole chromosome missegregation, and chromosome-specific cell viability. Using this model, we confirm the existence of an optimal rate of chromosome missegregation probabilities that maximizes karyotypic heterogeneity, while minimizing the occurrence of nullisomy. Interestingly, karyotypic heterogeneity is significantly more dependent on chromosome missegregation probabilities rather than the number of cell divisions, so that maximal heterogeneity can be reached rapidly (within a few hundred generations of cell division) at chromosome missegregation rates commonly observed in cancer cell lines. Conversely, at low missegregation rates, heterogeneity is constrained even after thousands of cell division events. This leads us to conclude that chromosome copy number heterogeneity is primarily constrained by chromosome missegregation rates and the risk for nullisomy and less so by the age of the tumor. This model enables direct integration of karyotype information into existing models of tumor evolution based on somatic mutations.

34. **The number of cycles with a given descent set** (with J. Troyka), *Proceedings of FPSAC 2018, Sém. Lothar. Combin.* 80B (2018), #8, 12 pp.

    Using a result of Gessel and Reutenauer, we find a simple formula for the number of cyclic permutations with a given descent set, by expressing it in terms of ordinary descent numbers (i.e., those counting all permutations with a given descent set). We then use this formula to show that, for almost all sets $I \subseteq [n-1]$, the fraction of size-n permutations with descent set I which are n-cycles is asymptotically $1/n$. As a special case, we recover a result of Stanley for alternating cycles. We also use our formula to count n-cycles with no two consecutive descents.

35. **Cyclic descent extensions and distributions** (with R. Adin, V. Reiner and Y. Roichman), *Proceedings of the GASCom 2018 Workshop*, 32–42.

    The notion of descent set is classical both for permutations and for standard Young tableaux (SYT). Cellini introduced a natural notion of cyclic descent set for permutations, and Rhoades introduced such a notion for SYT, but only of rectangular shapes. In this paper, we describe cyclic descents for SYT of various other shapes. Motivated by these findings, we define cyclic extensions of descent sets in a general context, and we show that they exist for SYT of almost all shapes. Finally, we introduce the ring of cyclic quasisymmetric functions and apply it to analyze the distributions of cyclic descents over permutations and SYT.

36. **Wilf equivalence relations for consecutive patterns** (with T. Dwyer), *Adv. in Appl. Math.* 99 (2018), 134–157.

    Two permutations π and τ are *c-Wilf equivalent* if, for each n, the number of permutations in $\mathcal{S}_n$ avoiding π as a consecutive pattern (i.e., in adjacent positions) is the same as the number of those avoiding τ. In addition, π and τ are *strongly c-Wilf equivalent* if, for each n and k, the number of permutations in $\mathcal{S}_n$ containing k occurrences of π as a consecutive pattern is the same as for τ. In this paper we introduce a third, more restrictive equivalence relation, defining π and τ to be *super-strongly c-Wilf equivalent* if the above condition holds for any set of prescribed positions for the k occurrences. We show that, when restricted to non-overlapping permutations, these three equivalence relations coincide.

    We also give a necessary condition for two permutations to be strongly c-Wilf equivalent. Specifically, we show that if $\pi, \tau \in \mathcal{S}_m$ are strongly c-Wilf equivalent, then $|\pi_m - \pi_1| = |\tau_m - \tau_1|$. In the special case of non-overlapping permutations π and τ, this proves a weaker version of a conjecture of the second author stating that π and τ are c-Wilf equivalent if and only if $\pi_1 = \tau_1$ and $\pi_m = \tau_m$, up to trivial symmetries. Finally, we strengthen a recent result of Nakamura and Khoroshkin–Shapiro giving sufficient conditions for strong c-Wilf equivalence.

37. **A necessary condition for c-Wilf equivalence** (with T. Dwyer), *Proceedings of FPSAC 2018, Sém. Lothar. Combin.* 78B (2017), #69, 12 pp.

38. **A bijection between bargraphs and Dyck paths** (with E. Deutsch), *Discrete Appl. Math.* 251 (2018), 340–344.

    Bargraphs are a special class of column-convex polyominoes. They can be identified with lattice paths with unit steps north, east, and south that start at the origin, end on the x-axis, and stay strictly above the x-axis everywhere except at the endpoints. Bargraphs, which are used to represent histograms and to model polymers in statistical physics, have been enumerated in the literature by semiperimeter and by several other statistics, using different methods such as the wasp-waist decomposition of Feretić, and a bijection with certain Motzkin paths.

    In this paper we describe an unusual bijection between bargraphs and Dyck paths, and study how some statistics are mapped by the bijection. As a consequence, we obtain a new interpretation of Catalan numbers, as counting bargraphs where the semiperimeter minus the number of peaks is fixed.

39. **Continued fractions for permutation statistics**, *Discrete Math. Theor. Comput. Sci.* 19 (2018), #11.

    We explore a bijection between permutations and colored Motzkin paths that has been used in different forms by Foata and Zeilberger, Biane, and Corteel. By giving a visual representation of this bijection in terms of so-called cycle diagrams, we find simple translations of some statistics on permutations (and subsets of permutations) into statistics on colored Motzkin paths, which are amenable to the use of continued fractions. We obtain new enumeration formulas for subsets of permutations with respect to fixed points, excedances, double excedances, cycles, and inversions. In particular, we prove that cyclic permutations whose excedances are increasing are counted by the Bell numbers.

40. **The probability of avoiding consecutive patterns in the Mallows distribution** (with H. Crane and S. DeSalvo), *Random Structures Algorithms* 53 (2018), 417–447.

    We use combinatorial and probabilistic techniques to study growth rates for the probability that a random permutation from the Mallows distribution avoids consecutive patterns. The Mallows distribution is a q-analogue of the uniform distribution weighting each permutation π by $q^{\mathrm{inv}(\pi)}$, where $\mathrm{inv}(\pi)$ is the number of inversions in π and q is a positive, real-valued parameter. We prove that the growth rate exists for all patterns and all $q > 0$, and we generalize Goulden and Jackson's cluster method to keep track of the number of inversions in permutations avoiding a given consecutive pattern. Using singularity analysis, we approximate the growth rates for length-3 patterns, monotone patterns, and non-overlapping patterns starting with 1, and we compare growth rates between different patterns. We also use Stein's method to show that, under certain assumptions on q and σ, the number of occurrences of a given pattern σ is well approximated by the normal distribution.

41. **Patterns of negative shifts and signed shifts** (with K. Archer and K. Moore), *Proceedings of FPSAC 2017, Sém. Lothar. Combin.* 78B (2017), #49, 12 pp.

    Given a function f from a linearly ordered set X to itself, we say that a permutation π is an *allowed pattern* of f if the relative order of the first n iterates of f beginning at some $x \in X$ is given by π. We give a characterization of the allowed patterns of signed shifts in terms of monotone runs of a certain transformation of π, which completes and simplifies the original characterization given by Amigó. Signed shifts, which are generalizations of the shift map where some slopes are allowed to be negative, are particularly well-suited to a combinatorial analysis. In the special case where all the slopes are negative, we give an exact formula for the number of allowed patterns. Finally, we obtain a combinatorial derivation of the topological entropy of signed shifts.

42. **On rotated Schur-positive sets** (with Y. Roichman), *J. Combin. Theory Ser. A* 152 (2017), 121–137.

The problem of finding Schur-positive sets of permutations, originally posed by Gessel and Reutenauer, has seen some recent developments. Schur-positive sets of pattern-avoiding permutations have been found by Sagan et al and a general construction based on geometric operations on grid classes has been given by the authors. In this paper we prove that horizontal rotations of Schur-positive subsets of permutations are always Schur-positive. The proof applies a cyclic action on standard Young tableaux of certain skew shapes and a *jeu-de-taquin* type straightening algorithm. As a consequence of the proof we obtain a notion of cyclic descent set on these tableaux, which is rotated by the cyclic action on them.

43. **Character formulas and descents for the hyperoctahedral group** (with R. Adin, C. Athanasiadis and Y. Roichman), *Adv. in Appl. Math.* 87 (2017), 128–169.

A general setting to study a certain type of formulas, expressing characters of the symmetric group $\mathfrak{S}_n$ explicitly in terms of descent sets of combinatorial objects, has been developed by two of the authors. This theory is further investigated in this paper and extended to the hyperoctahedral group B_n. Key ingredients are a new formula for the irreducible characters of B_n, the signed quasisymmetric functions introduced by Poirier, and a new family of matrices of Walsh–Hadamard type. Applications include formulas for natural B_n-actions on coinvariant and exterior algebras and on the top homology of a certain poset in terms of the combinatorics of various classes of signed permutations, as well as a B_n-analogue of an equidistribution theorem of Désarménien and Wachs.

44. **The structure of the consecutive pattern poset** (with P. McNamara), *Int. Math. Res. Not. IMRN* 2018, no. 7, 2099–2134.

The consecutive pattern poset is the infinite partially ordered set of all permutations where $\sigma \leq \tau$ if τ has a subsequence of adjacent entries in the same relative order as the entries of σ. We study the structure of the intervals in this poset from topological, poset-theoretic, and enumerative perspectives. In particular, we prove that all intervals are rank-unimodal and strongly Sperner, and we characterize disconnected and shellable intervals. We also show that most intervals are not shellable and have Möbius function equal to zero.

45. **On intervals of the consecutive pattern poset** (with P. McNamara), *Discrete Math. Theor. Comput. Sci. proc. BC* (2016), 431–442.

46. **Statistics on bargraphs viewed as cornerless Motzkin paths** (with E. Deutsch), *Discrete Appl. Math.* 221 (2017), 54–66.

A bargraph is a self-avoiding lattice path with steps $U = (0,1)$, $H = (1,0)$ and $D = (0,-1)$ that starts at the origin and ends on the x-axis, and stays strictly above the x-axis everywhere except at the endpoints. Bargraphs have been studied as a special class of convex polyominoes, and enumerated using the so-called wasp-waist decomposition of Bousquet-Mélou and Rechnitzer. In this paper we note that there is a trivial bijection between bargraphs and Motzkin paths without peaks or valleys. This allows us to use the recursive structure of Motzkin paths to enumerate bargraphs with respect to several statistics, finding simpler derivations of known results and obtaining many new ones. We also count symmetric bargraphs and alternating bargraphs. In some cases we construct statistic-preserving bijections between different combinatorial objects, proving some identities that we encounter along the way.

47. **Schur-positive sets of permutations via products of grid classes** (with Y. Roichman), *J. Algebraic Combin.* 45 (2017), 363–405.

Characterizing sets of permutations whose associated quasisymmetric function is symmetric and Schur-positive is a long-standing problem in algebraic combinatorics. In this paper we present a general method to construct Schur-positive sets and multisets, based on geometric grid classes and the product operation. Our approach produces many new instances of Schur-positive sets, and provides a broad framework that explains the existence of known such sets that until now were sporadic cases.

48. **Schur-positivity via products of grid classes** (with Y. Roichman), *Discrete Math. Theor. Comput. Sci. proc. BC* (2016), 443–454.

49. **Two descent statistics over 321-avoiding centrosymmetric involutions** (with M. Barnabei, F. Bonetti, M. Silimbani), *Electron. J. Combin.*, 23(1) (2016), #P1.35.

    Centrosymmetric involutions in the symmetric group $\mathcal{S}_{2n}$ are permutations π such that $\pi = \pi^{-1}$ and $\pi(i) + \pi(2n+1-i) = 2n+1$ for all i, and they are in bijection with involutions of the hyperoctahedral group. We describe the distribution of some natural descent statistics on 321-avoiding centrosymmetric involutions, including the number of descents in the first half of the involution, and the sum of the positions of these descents. Our results are based on two new bijections, one between centrosymmetric involutions in $\mathcal{S}_{2n}$ and subsets of $\{1,\dots,n\}$, and another one showing that certain statistics on Young diagrams that fit inside a rectangle are equidistributed. We also use the latter bijection to refine a known result stating that the distribution of the major index on 321-avoiding involutions is given by the q-analogue of the central binomial coefficients.

50. **Symmetries of statistics on lattice paths between two boundaries** (with M. Rubey), *Adv. Math.* 287 (2016), 347–388.

    We prove that on the set of lattice paths with steps $N = (0, 1)$ and $E = (1, 0)$ that lie between two fixed boundaries T and B (which are themselves lattice paths), the statistics 'number of E steps shared with B' and 'number of E steps shared with T' have a symmetric joint distribution. To do so, we give an involution that switches these statistics, preserves additional parameters, and generalizes to paths that contain steps $S = (0, -1)$ at prescribed x-coordinates. We also show that a similar equidistribution result for path statistics follows from the fact that the Tutte polynomial of a matroid is independent of the order of its ground set. We extend the two theorems to k-tuples of paths between two boundaries, and we give some applications to Dyck paths, generalizing a result of Deutsch, to watermelon configurations, to pattern-avoiding permutations, and to the generalized Tamari lattice.

    Finally, we prove a conjecture of Nicolás about the distribution of degrees of k consecutive vertices in k-triangulations of a convex n-gon. To achieve this goal, we provide a new statistic-preserving bijection between certain k-tuples of non-crossing paths and k-flagged semistandard Young tableaux, which is based on local moves reminiscent of *jeu de taquin*.

51. **Bijections for lattice paths between two boundaries** (with M. Rubey), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 827–838.

52. **Dynamics of tumor heterogeneity derived from clonal karyotypic evolution** (with A. Laughney, S. Backhoum and G. Genovese), *Cell Rep.* 12 (2015), 809–820.

    Numerical chromosomal instability is a ubiquitous feature of human neoplasms. Due to experimental limitations, fundamental characteristics of karyotypic changes in cancer are poorly understood. Using an experimentally inspired stochastic model, based on the potency and chromosomal distribution of oncogenes and tumor suppressor genes, we show that cancer cells evolved to exist within a narrow range of chromosome missegregation rates that optimizes phenotypic heterogeneity and clonal survival. Departure from this range reduces clonal fitness and limits subclonal diversity. Mapping of the aneuploid fitness landscape reveals a highly favorable, commonly observed, near-triploid state onto which evolving diploid and tetraploid-derived populations spontaneously converge, albeit at a much lower fitness cost for the latter. Finally, by analyzing 1,368 chromosomal translocation events in five human cancers, we find that karyotypic evolution also shapes chromosomal translocation patterns by selecting for more oncogenic derivative chromosomes. Thus, chromosomal instability can generate the heterogeneity required for Darwinian tumor evolution.

53. **A survey of consecutive patterns in permutations**, chapter of the book *Recent Trends in Combinatorics (IMA Volume in Mathematics and its Applications)* 601–618, Springer 2016.

    A consecutive pattern in a permutation π is another permutation σ determined by the relative order of a subsequence of contiguous entries of π. Traditional notions such as descents, runs and peaks can be viewed as particular examples of consecutive patterns in permutations, but the systematic study of these patterns has flourished in the last 15 years, during which a variety of different techniques have been used. We survey some interesting developments in the subject, focusing on exact and asymptotic enumeration results, the classification of consecutive patterns into equivalence classes, and their applications to the study of one-dimensional dynamical systems.

54. **The frequency of pattern occurrence in random walks** (with M. Martinez), *Discrete Math. Theor. Comput. Sci. proc.*, to appear.

    In the past decade, the use of ordinal patterns in the analysis of time series and dynamical systems has become an important tool. Ordinal patterns (otherwise known as a permutation patterns) are found in time series by taking n data points at evenly-spaced time intervals and mapping them to a length-n permutation determined by relative ordering. The frequency with which certain patterns occur is a useful statistic for such series. However, the behavior of the frequency of pattern occurrence is unstudied for most models. We look at the frequency of pattern occurrence in random walks in discrete time, and we define a natural equivalence relation on permutations under which equivalent patterns appear with equal frequency, regardless of probability distribution. We characterize these equivalence classes applying combinatorial methods.

55. **Bijections for pairs of non-crossing lattice paths and walks in the plane**, *European J. Combin.* 49 (2015), 25–41.

    It is a classical result in combinatorics that among lattice paths with $2m$ steps $U = (1,1)$ and $D = (1,-1)$ starting at the origin, the number of those that do not go below the x-axis equals the number of those that end on the x-axis. A much more unfamiliar fact is that the analogous equality obtained by replacing single paths with k-tuples of non-crossing paths holds for every k. This result has appeared in the literature in different contexts involving plane partitions (where it was proved by Proctor), partially ordered sets, Young tableaux, and lattice walks, but no bijective proof for $k \geq 2$ seems to be known.

    In this paper we give a bijective proof of the equality for $k = 2$, showing that for pairs of non-crossing lattice paths with $2m$ steps U and D, the number of those that do not go below the x-axis equals the number of those that end on the x-axis. Translated in terms of walks in the plane starting at the origin with $2m$ unit steps in the four coordinate directions, our work provides correspondences among those constrained to the first octant, those constrained to the first quadrant that end on the x-axis, and those in the upper half-plane that end at the origin.

    Our bijections, which are defined in more generality, also prove new results where different endpoints are allowed, and they give a bijective proof of the formula for the number of walks in the first octant that end on the diagonal, partially answering a question of Bousquet-Mélou and Mishna.

56. **Signed arc permutations** (with Y. Roichman), *J. Comb.* 6 (2015), 205–234.

    Arc permutations, which were originally introduced in the study of triangulations and characters, have recently been shown to have interesting combinatorial properties. The first part of this paper continues their study by providing signed enumeration formulas with respect to their descent set and major index. Next, we generalize the notion of arc permutations to the hyperoctahedral group in two different directions. We show that these extensions to type B carry interesting analogues of the properties of type A arc permutations, such as characterizations by pattern avoidance, and elegant unsigned and signed enumeration formulas with respect to the flag-major index.

57. **A generating tree approach to k-nonnesting partitions and permutations** (with M. Mishna, S. Burrill and L. Yen), *Ann. Comb.* 20 (2016), 453–485.

    We describe a generating tree approach to the enumeration and exhaustive generation of k-nonnesting set partitions and permutations. Unlike previous work in the literature using the connections of these objects to Young tableaux and restricted lattice walks, our approach deals directly with partition and permutation diagrams. We provide explicit functional equations for the generating functions, with k as a parameter. Key to the solution is a superset of diagrams that permit semi-arcs. Many of the resulting counting sequences also count other well known objects, such as Baxter permutations, and Young tableaux of bounded height.

58. **Generating trees for partitions and permutations with no k-nestings** (with S. Burrill, M. Mishna, L. Yen), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 409–420.

59. **Descent sets on 321-avoiding involutions and hook decompositions of partitions** (with M. Barnabei, F. Bonetti and M. Silimbani), *J. Combin. Theory Ser. A* 128 (2014), 132–148.

    We show that the distribution of the major index over the set of involutions in $\mathcal{S}_n$ that avoid the pattern 321 is given by the q-analogue of the n-th central binomial coefficient. The proof consists of a composition of three non-trivial bijections, one being the Robinson-Schensted correspondence, ultimately mapping those involutions with major index m into partitions of m whose Young diagram fits inside a $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ box. We also obtain a refinement that keeps track of the descent set, and we deduce an analogous result for the comajor index of 123-avoiding involutions.

60. **Cyclic permutations realized by signed shifts** (with K. Archer), *J. Comb.* 5 (2014), 1–30.

    The periodic (ordinal) patterns of a map are the permutations realized by the relative order of the points in its periodic orbits. We give a combinatorial characterization of the periodic patterns of an arbitrary signed shift, in terms of the structure of the descent set of a certain cyclic permutation associated to the pattern. Signed shifts are an important family of one-dimensional dynamical systems that includes shift maps and the tent map as particular cases. Defined as a function on the set of infinite words on a finite alphabet, a signed shift deletes the first letter and, depending on its value, possibly applies the complementation operation on the remaining word. For shift maps, reverse shift maps, and the tent map, we give exact formulas for their number of periodic patterns. As a byproduct of our work, we recover results of Gessel–Reutenauer and Weiss–Rogers and obtain new enumeration formulas for pattern-avoiding cycles.

61. **Periodic patterns of signed shifts** (with K. Archer), *Discrete Math. Theor. Comput. Sci. proc. AS* (2013), 873–884.

62. **Inversion polynomials for 321-avoiding permutations** (with S.-E. Cheng, A. Kasraoui and B. Sagan), *Discrete Math.* 313 (2013), 2552–2565.

    We prove a generalization of a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor giving a recursion for the inversion polynomial of 321-avoiding permutations. We also answer a question they posed about finding a recursive formula for the major index polynomial of 321-avoiding permutations. Other properties of these polynomials are investigated as well. Our tools include Dyck and 2-Motzkin paths, polyominoes, and continued fractions.

63. **Arc permutations** (with Y. Roichman), *J. Algebraic Combin.* 39 (2014), 301–334.

Arc permutations and unimodal permutations were introduced in the study of triangulations and characters. In this paper we describe combinatorial properties of these permutations, including characterizations in terms of pattern avoidance, connections to Young tableaux, and an affine Weyl group action on them.

64. **Arc Permutations (extended abstract)** (with Y. Roichman), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 259–270.

65. **Pattern avoidance in matchings and partitions** (with J. Bloom), *Electron. J. Combin.* 20 (2013), #P5.

Extending the notion of pattern avoidance in permutations, we study matchings and set partitions whose arc diagram representation avoids a given configuration of three arcs. These configurations, which generalize 3-crossings and 3-nestings, have an interpretation, in the case of matchings, in terms of patterns in full rook placements on Ferrers boards.

We enumerate 312-avoiding matchings and partitions, obtaining algebraic generating functions, in contrast with the known D-finite generating functions for the 321-avoiding (i.e., 3-noncrossing) case. Our approach provides a more direct proof of a formula of Bóna for the number of 1342-avoiding permutations. We also give a bijective proof of the shape-Wilf-equivalence of the patterns 321 and 213 which greatly simplifies existing proofs by Backelin–West–Xin and Jelínek, and provides an extension of work of Gouyou-Beauchamps for matchings with fixed points. Finally, we classify pairs of patterns of length 3 according to shape-Wilf-equivalence, and enumerate matchings and partitions avoiding a pair in most of the resulting equivalence classes.

66. **Patterns in matchings and rook placements** (with J. Bloom), *Discrete Math. Theor. Comput. Sci. proc. AS* (2013), 909–920.

67. **The most and the least avoided consecutive patterns**, *Proc. Lond. Math. Soc.* 106 (2013), 957–979.

We prove that the number of permutations avoiding the consecutive pattern $12\dots m$ —that is, containing no m adjacent entries in increasing order— is asymptotically larger than the number of permutations avoiding any other consecutive pattern of length m. This settles a conjecture of the author and Noy from 2001. We also prove a recent conjecture of Nakamura stating that, at the other end of the spectrum, the number of permutations avoiding $12\dots(m-2)m(m-1)$ is asymptotically smaller than for any other pattern. Finally, we consider non-overlapping patterns and obtain analogous results describing the most and least avoided ones.

The techniques used include the cluster method of Goulden and Jackson, an interpretation of clusters as linear extensions of posets, and singularity analysis of generating functions.

68. **Clusters, generating functions and asymptotics for consecutive patterns in permutations** (with M. Noy), *Adv. in Appl. Math.* 49 (2012) 351–374.

We use the cluster method to enumerate permutations avoiding consecutive patterns. We reprove and generalize in a unified way several known results and obtain new ones, including some patterns of length 4 and 5, as well as some infinite families of patterns of a given shape. By enumerating linear extensions of certain posets, we find a differential equation satisfied by the inverse of the exponential generating function counting occurrences of the pattern. We prove that for a large class of patterns, this inverse is always an entire function.

We also complete the classification of consecutive patterns of length up to 6 into equivalence classes, proving a conjecture of Nakamura. Finally, we show that the monotone pattern asymptotically dominates (in the sense that it is easiest to avoid) all non-overlapping patterns of the same length, thus proving a conjecture of Elizalde and Noy for a positive fraction of all patterns.

69. **Consecutive patterns in permutations: clusters and generating functions** (with M. Noy), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 247–258.

70. **Total occurrence statistics on restricted permutations** (with A. Burstein), *PU. M. A. Pure Math. Appl. (PU.M.A.)* 24 (2013), 103–123.

    We study the total number of occurrences of several vincular (also called generalized) patterns and other statistics, such as the major index and the Denert statistic, on permutations avoiding a pattern of length 3, extending results of Bóna and Homberger. In particular, for $2-3-1$-avoiding permutations, we find the total number of occurrences of any vincular pattern of length 3. In some cases the answer is given by simple expressions involving binomial coefficients. The tools we use are bijections with Dyck paths, generating functions, and block decompositions of permutations.

71. **Permutations and β-shifts**, *J. Combin. Theory Ser. A* 118 (2011), 2474–2497.

    Given a real number $\beta > 1$, a permutation π of length n is realized by the β-shift if there is some $x \in [0,1]$ such that the relative order of the sequence $x, f(x), \dots, f^{n-1}(x)$, where $f(x)$ is the fractional part of βx, is the same as that of the entries of π. Widely studied from such diverse fields as number theory and automata theory, β-shifts are prototypical examples of one-dimensional chaotic dynamical systems. When β is an integer, permutations realized by shifts were studied in [*SIAM J. Discrete Math.* 23 (2009), 765–786]. In this paper we generalize some of the results to arbitrary β-shifts. We describe a method to compute, for any given permutation π, the smallest β such that π is realized by the β-shift. We also give a way to determine the length of the shortest forbidden (i.e., not realized) pattern of an arbitrary β-shift.

72. **Allowed patterns of β-shifts**, *Discrete Math. Theor. Comput. Sci. proc. AO* (2011), 293–304.

73. **Descent sets of cyclic permutations**, *Adv. in Appl. Math.* 47 (2011), 688–709.

    We present a bijection between cyclic permutations of $\{1,2,\dots,n+1\}$ and permutations of $\{1,2,\dots,n\}$ that preserves the descent set of the first n entries and the set of weak excedances. This non-trivial bijection involves a Foata-like transformation on the cyclic notation of the permutation, followed by certain conjugations. We also give an alternate derivation of the consequent result about the equidistribution of descent sets using work of Gessel and Reutenauer. Finally, we prove a conjecture of the author in [*SIAM J. Discrete Math.* 23 (2009), 765–786] and a conjecture of Eriksen, Freij and Wästlund.

74. **Restricted simsun permutations** (with E. Deutsch), *Ann. Comb.* 16 (2012), 253–269.

    A permutation is simsun if for all k, the subword of the one-line notation consisting of the k smallest entries does not have three consecutive decreasing elements. Simsun permutations were introduced by Simion and Sundaram, who showed that they are counted by the Euler numbers. In this paper we enumerate simsun permutations avoiding a pattern or a set of patterns of length 3. The results involve Motkzin, Fibonacci, and secondary structure numbers. The techniques in the proofs include generating functions, bijections into lattice paths and generating trees.

75. **On basic forbidden patterns of functions** (with Y. Liu), *Discrete Appl. Math.* 159 (2011), 1207–1216.

    The allowed patterns of a map on a one-dimensional interval are those permutations that are realized by the relative order of the elements in its orbits. The set of allowed patterns is completely determined by the minimal patterns that are not allowed. These are called basic forbidden patterns. In this paper we study basic forbidden patterns of several functions. We show that the logistic map $L_r(x) = rx(1-x)$ and some generalizations have infinitely many of them for $1 < r \leq 4$, and we give a lower bound on the number of basic forbidden patterns of L_4 of each length. Next, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map. Finally, we provide some necessary conditions for a set of permutations to be the set of basic forbidden patterns of such a map.

76. **Cycle-up-down permutations** (with E. Deutsch), *Australas. J. Combin.* 50 (2011), 187–199.

    A permutation is defined to be *cycle-up-down* if it is a product of cycles that, when written starting with their smallest element, have an up-down pattern. We prove bijectively and analytically that these permutations are enumerated by the Euler numbers, and we study the distribution of some statistics on them, as well as on up-down permutations and on all permutations. The statistics include the number of cycles of even and odd length, the number of left-to-right minima, and the number of extreme elements.

77. **The X-class and almost-increasing permutations**, *Ann. Comb.* 15 (2011), 51–68.

    In this paper we give a bijection between the class of permutations that can be drawn on an X-shape and a certain set of permutations that appears in [Knuth, The Art of Computer Programming, Vol. III], in connection to sorting algorithms. A natural generalization of this set leads us to the definition of *almost-increasing permutations*, which is a one-parameter family of permutations that can be characterized in terms of forbidden patterns. We find generating functions for almost-increasing permutations by using their cycle structure to map them to colored Motzkin paths. We also give refined enumerations with respect to the number of cycles, fixed points, excedances, and inversions.

78. **Improved bounds on the number of numerical semigroups of a given genus**, *J. Pure Appl. Algebra* 214 (2010), 1862–1873.

    We improve the previously best known lower and upper bounds on the number n_g of numerical semigroups of genus g. Starting from a known recursive description of the tree $\mathcal{T}$ of numerical semigroups, we analyze some of its properties and use them to construct approximations of $\mathcal{T}$ by generating trees whose nodes are labeled by certain parameters of the semigroups. We then translate the succession rules of these trees into functional equations for the generating functions that enumerate their nodes, and solve these equations to obtain the bounds. Some of our bounds involve the Fibonacci numbers, and the others are expressed as generating functions.

    We also give upper bounds on the number of numerical semigroups having an infinite number of descendants in $\mathcal{T}$.

79. **The largest and the smallest fixed points of permutations** (with E. Deutsch), *European J. Combin.* 31 (2010), 1404–1409.

    We give a new interpretation of the derangement numbers d_n as the sum of the values of the largest fixed points of all non-derangements of length $n-1$. We also show that the analogous sum for the smallest fixed points equals the number of permutations of length n with at least two fixed points. We provide analytic and bijective proofs of both results, as well as a new recurrence for the derangement numbers.

80. **The number of permutations realized by a shift**, *SIAM J. Discrete Math.* 23 (2009), 765–786.

    A permutation π is realized by the shift on N symbols if there is an infinite word on an N-letter alphabet whose successive left shifts by one position are lexicographically in the same relative order as π. The set of realized permutations is closed under consecutive pattern containment. Permutations that cannot be realized are called forbidden patterns. It was shown by Amigó–Elizalde–Kennel that the shortest forbidden patterns of the shift on N symbols have length $N+2$. In this paper we give a characterization of the set of permutations that are realized by the shift on N symbols, and we enumerate them according to their length.

81. **Permutations realized by shifts**, *Discrete Math. Theor. Comput. Sci. proc. AK* (2009), 361–372.

82. **Sorting by Placement and Shift** (with P. Winkler), *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2009*, `arXiv:0809.2957v1`.

    In sorting situations where the final destination of each item is known, it is natural to repeatedly choose items and place them where they belong, allowing the intervening items to shift by one to make room. (In fact, a special case of this algorithm is commonly used to hand-sort files.) However, it is not obvious that this algorithm necessarily terminates.

    We show that in fact the algorithm terminates after at most $2^{n-1}-1$ steps in the worst case (confirming a conjecture of L. Larson), and that there are super-exponentially many permutations for which this exact bound can be achieved. The proof involves a curious symmetrical binary representation.

83. **Forbidden patterns and shift systems** (with J.M. Amigó, M.B. Kennel), *J. Combin. Theory Ser. A*, 115 (2008), 485–504.

    The scope of this paper is two-fold. First, to present an interesting implementation of permutations avoiding generalized patterns in the framework of discrete-time dynamical systems. Indeed, the orbits generated by piecewise monotone maps on one-dimensional intervals have forbidden order patterns, which do not occur in any orbit. The allowed patterns are then those patterns avoiding the so-called forbidden root patterns and their translates. The second scope is to study forbidden patterns in shift systems, which are universal models in information theory, dynamical systems and stochastic processes. Due to its simple structure, shift systems are accessible to a more detailed analysis and, at the same time, exhibit all important properties of complex dynamical systems, allowing to export the results to other dynamical systems via order-isomorphisms.

84. **Pattern avoidance in dynamical systems** (with J.M. Amigó, M.B. Kennel), *Discrete Math. Theor. Comput. Sci. proc. AJ* (2008), 71–82.

85. **Generating trees for permutations avoiding generalized patterns**, *Ann. Comb.* 11 (2007), 435–458.

    We construct generating trees with with one and two labels for some classes of permutations avoiding generalized patterns of length 3 and 4. These trees are built by adding at each level an entry to the right end of the permutation, instead of inserting always the largest entry. This allows us to incorporate the adjacency condition about some entries in an occurrence of a generalized pattern. We find functional equations for the generating functions enumerating these classes of permutations with respect to different parameters, and in a few cases we solve them using some techniques of Bousquet-Mélou, recovering known enumerative results and finding new ones.

86. **A bijection between 2-triangulations and pairs of non-crossing Dyck paths**, *J. Combin. Theory Ser. A*, 114/8 (2007), 1481–1503.

    A k-triangulation of a convex polygon is a maximal set of diagonals so that no $k+1$ of them mutually cross in their interiors. We present a bijection between 2-triangulations of a convex n-gon and pairs of non-crossing Dyck paths of length $2(n-4)$. This solves the problem of finding a bijective proof of a result of Jonsson for the case $k=2$. We obtain the bijection by constructing isomorphic generating trees for the sets of 2-triangulations and pairs of non-crossing Dyck paths.

87. **Bounds on the number of inference functions of a graphical model** (with K. Woods), *Statist. Sinica* 17 (2007), 1395–1415.

    We give an upper bound on the number of inference functions of any directed graphical model. This bound is polynomial on the size of the model, for a fixed number of parameters, thus improving the exponential upper bound given by Pachter and Sturmfels. We also show that our bound is tight up to a constant factor, by constructing a family of hidden Markov models whose number of inference functions agrees asymptotically with the upper bound. Finally, we apply this bound to a model for sequence alignment that is used in computational biology.

88. **The probability of choosing primitive sets** (with K. Woods), *J. Number Theory* 125 (2007), 39–49.

    We generalize a theorem of Nymann that the density of points in $\mathbb{Z}^d$ that are visible from the origin is $1/\zeta(d)$, where $\zeta(a)$ is the Riemann zeta function $\sum_{i=1}^{\infty} 1/i^a$. A subset $S \subset \mathbb{Z}^d$ is called primitive if it is a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^d \cap span_{\mathbb{R}}(S)$, or, equivalently, if S can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. We prove that if m points in $\mathbb{Z}^d$ are chosen uniformly and independently at random from a large box, then as the size of the box goes to infinity, the probability that the points form a primitive set approaches $1/(\zeta(d)\zeta(d-1)\cdots\zeta(d-m+1))$.

89. **Restricted Dumont permutations, Dyck paths, and noncrossing partitions** (with A. Burstein and T. Mansour), *Discrete Math.* 306 (2006), 2851–2869.

    We complete the enumeration of Dumont permutations of the second kind avoiding a pattern of length 4 which is itself a Dumont permutation of the second kind. We also consider some combinatorial statistics on Dumont permutations avoiding certain patterns of length 3 and 4 and give a natural bijection between 3142-avoiding Dumont permutations of the second kind and noncrossing partitions that uses cycle decomposition, as well as bijections between 132-, 231- and 321-avoiding Dumont permutations and Dyck paths. Finally, we enumerate Dumont permutations of the first kind simultaneously avoiding certain pairs of 4-letter patterns and another pattern of arbitrary length.

90. **Asymptotic enumeration of permutations avoiding generalized patterns**, *Adv. in Appl. Math.* 36 (2006), 138–155.

    Motivated by the recent proof of the Stanley-Wilf conjecture, we study the asymptotic behavior of the number of permutations avoiding a generalized pattern. Generalized patterns allow the requirement that some pairs of letters must be adjacent in an occurrence of the pattern in the permutation, and consecutive patterns are a particular case of them. We determine the asymptotic behavior of the number of permutations avoiding a consecutive pattern, showing that they are an exponentially small proportion of the total number of permutations. For some other generalized patterns we give partial results, showing that the number of permutations avoiding them grows faster than for classical patterns but more slowly than for consecutive patterns.

91. **Inference functions**, chapter of the book *Algebraic Statistics for Computational Biology*, edited by Lior Pachter and Bernd Sturmfels, Cambridge University Press, 2005.

92. **Bounds for optimal sequence alignment** (with Fumei Lam), chapter of the book *Algebraic Statistics for Computational Biology*, edited by Lior Pachter and Bernd Sturmfels, Cambridge University Press, 2005.

93. **Old and young leaves on plane trees** (with William Y.C. Chen and Emeric Deutsch), *European J. Combin.* 27 (2006), Issue 3, 414–427.

    A leaf of a plane tree is called an old leaf if it is the leftmost child of its parent, and it is called a young leaf otherwise. In this paper we enumerate plane trees with a given number of old leaves and young leaves. The formula is obtained combinatorially by presenting two bijections between plane trees and 2-Motzkin paths which map young leaves to red horizontal steps, and old leaves to up steps plus one. We derive some implications to the enumeration of restricted permutations with respect to certain statistics such as pairs of consecutive deficiencies, double descents, and ascending runs. Finally, our main bijection is applied to obtain refinements of two identities of Coker, involving refined Narayana numbers and the Catalan numbers.

94. **Multiple pattern avoidance with respect to fixed points and excedances**, *Electron. J. Combin.* 11 (2004), #R51.

We study the distribution of the statistics 'number of fixed points' and 'number of excedances' in permutations avoiding subsets of patterns of length 3. We solve all the cases of simultaneous avoidance of more than one pattern, giving generating functions enumerating these two statistics. Some cases are generalized to patterns of arbitrary length. For avoidance of one single pattern we give partial results. We also describe the distribution of these statistics in involutions avoiding any subset of patterns of length 3.
The main technique is to use bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that the statistics in permutations that we study correspond to statistics on Dyck paths that are easy to enumerate.

95. **Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials** (with Toufik Mansour), *Discrete Math.* 305 (2005), 170–189.

We say that a permutation π is a *Motzkin permutation* if it avoids 132 and there do not exist $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. We study the distribution of several statistics in Motzkin permutations, including the length of the longest increasing and decreasing subsequences and the number of rises and descents. We also enumerate Motzkin permutations with additional restrictions, and study the distribution of occurrences of fairly general patterns in this class of permutations.

96. **A simple and unusual bijection for Dyck paths and its consequences** (with Emeric Deutsch), *Ann. Comb.* 7 (2003), 281–297.

In this paper we introduce a new bijection from the set of Dyck paths to itself. This bijection has the property that it maps statistics that appeared recently in the study of pattern-avoiding permutations into classical statistics on Dyck paths, whose distribution is easy to obtain.
We also present a generalization of the bijection, as well as several applications of it to enumeration problems of statistics in restricted permutations.

97. **Bijections for refined restricted permutations** (with Igor Pak), *J. Combin. Theory Ser. A* 105 (2004), 207–219. *(This paper made it to the top three downloaded articles from JCTA in 2004.)*

We present a bijection between 321- and 132-avoiding permutations that preserves the number of fixed points and the number of excedances. This gives a simple combinatorial proof of recent results of Robertson, Saracino and Zeilberger [RSZ], and the first author [Eli]. We also show that our bijection preserves additional statistics, which extends the previous results.

98. **Fixed points and excedances in restricted permutations**, *Proceedings FPSAC 2003*, *Electron. J. Combin.* 18 (2012), #P29.

Using an unprecedented technique involving diagonals of non-rational generating functions, we prove that among the permutations of length n with i fixed points and j excedances, the number of 321-avoiding ones equals the number of 132-avoiding ones, for any given i, j.

This theorem generalizes a result of Robertson, Saracino and Zeilberger, for which we also give another, more direct proof.

99. **Consecutive patterns in permutations** (with Marc Noy), *Adv. in Appl. Math.* 30 (2003), 110–125.

In this paper we study the distribution of the number of occurrences of a permutation σ as a subword among all permutations in $\mathcal{S}_n$. We solve the problem in several cases depending on the shape of

σ by obtaining the corresponding bivariate exponential generating functions as solutions of certain linear differential equations with polynomial coefficients. Our method is based on the representation of permutations as increasing binary trees and on symbolic methods.

Others

100. **Combinatòria i biologia: funcions d'inferència i alineació de seqüències**, *Butl. Soc. Catalana Mat.* 21 (2006), n. 1, 39–52 (in Catalan).

101. **Statistics on pattern-avoiding permutations**, Ph.D. Thesis, MIT, 2004, available at `http://www.math.dartmouth.edu/~sergi/papers/thesis.pdf`.

102. **Games and Invariants**, chapter of the book **Training Sessions for the International Mathematical Olympiad** (in Catalan), Catalan Mathematical Society, Barcelona, 2000.