A bijection between 2-triangulations and pairs of non-crossing Dyck paths

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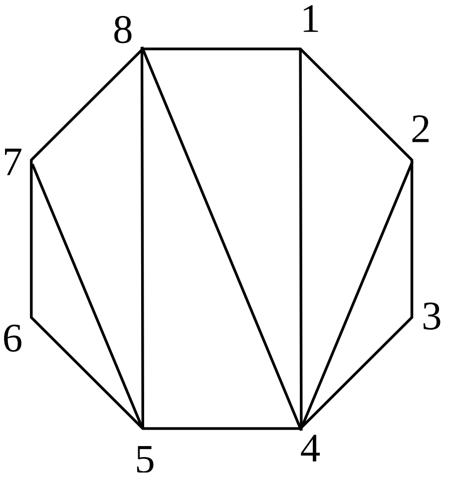
Let $\mathfrak{C}_n$ be a regular n-gon.
A triangulation of $\mathfrak{C}_n$ is a subdivision of $\mathfrak{C}_n$ into triangles, using diagonals that do not cross.

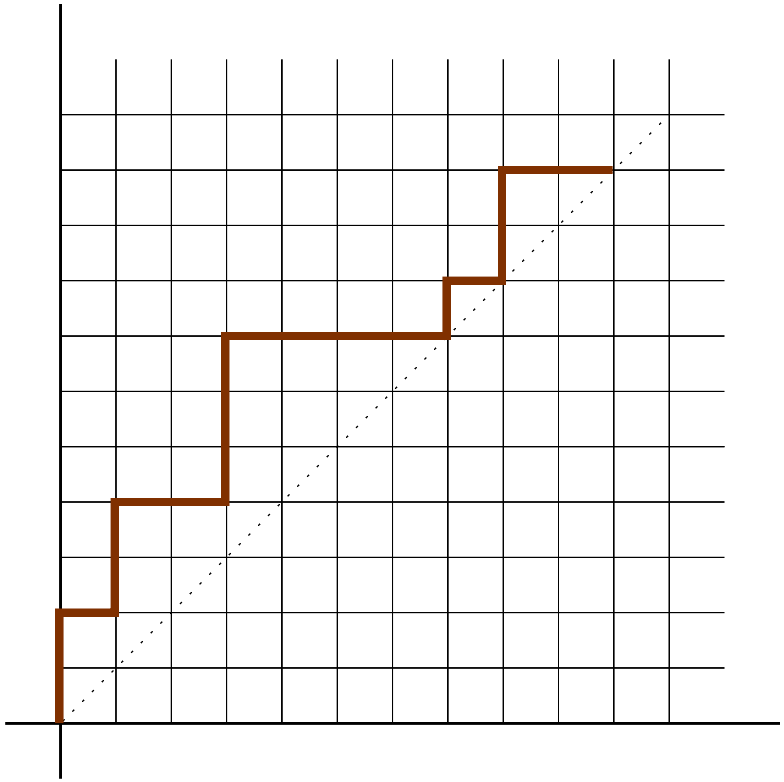


- Every triangulation of $\mathfrak{C}_n$ has exactly $n-3$ diagonals.
- The number of triangulations of $\mathfrak{C}_n$ is

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2},$$

where C_m is the m-th *Catalan number*.

A *Dyck path* is a lattice path from $(0, 0)$ to a point on the diagonal $y = x$ with steps $N = (0, 1)$ and $E = (1, 0)$ that never goes below this diagonal.



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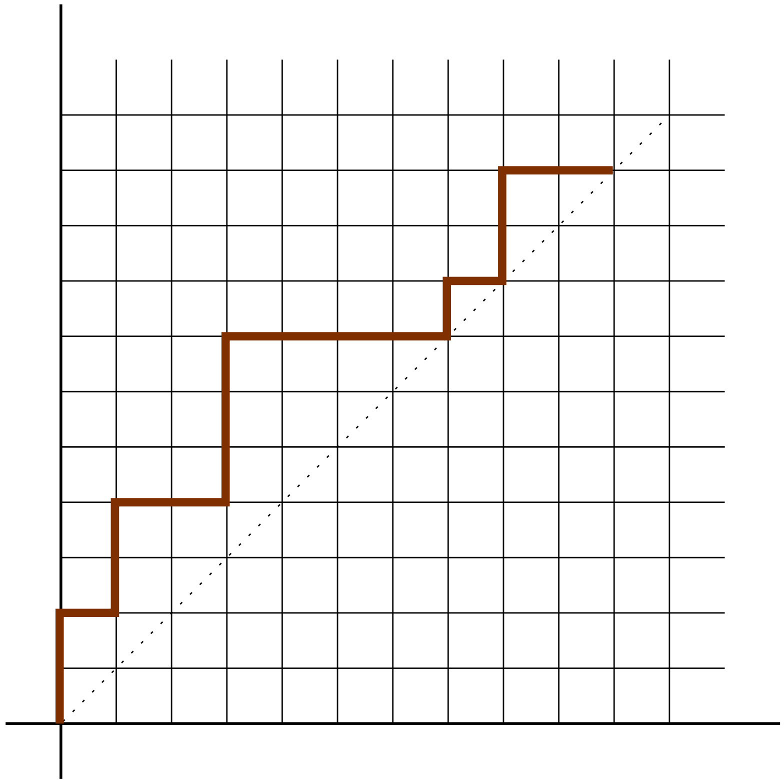
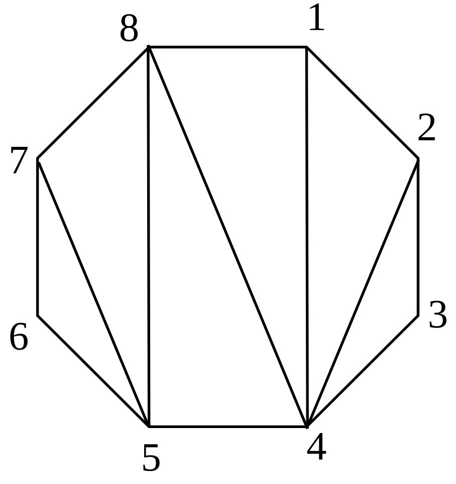
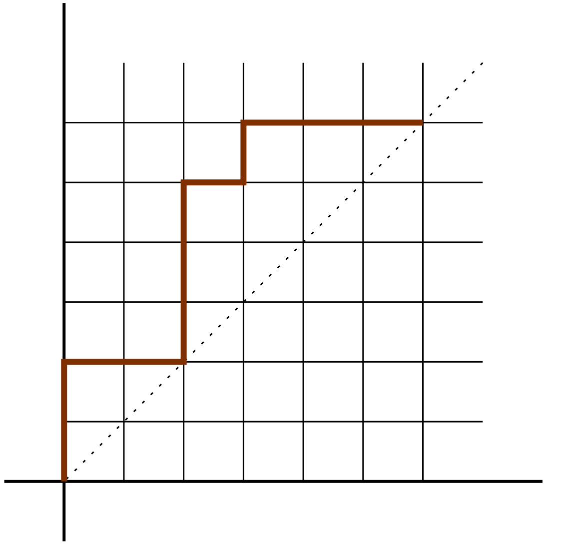


If (m,m) is the final point, we call m the *size* of the path.

Let $\mathcal{D}_m$ be the set of Dyck paths of size m. Then, $|\mathcal{D}_m| = C_m$.

A bijection between triangulations and Dyck paths

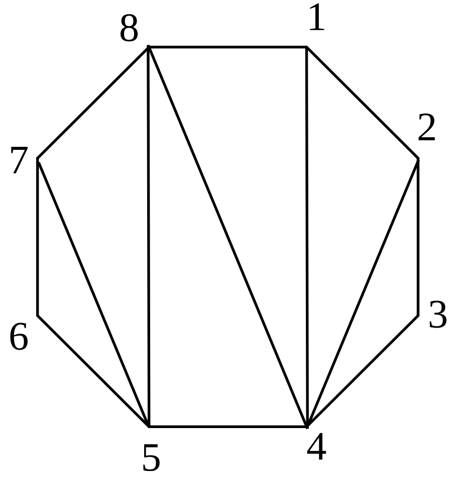


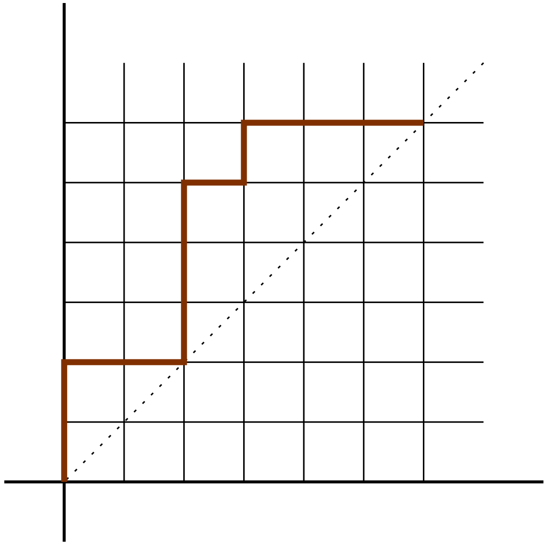


For each $j = 3, 4, \dots, n$:

- draw an N step,
- draw as many E steps as diagonals of the form (i, j) with $i < j$.

A bijection between triangulations and Dyck paths





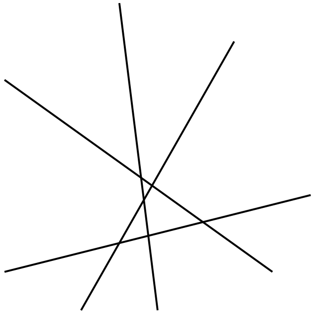
For each $j = 3, 4, \ldots, n$:

- draw an N step,
- draw as many E steps as diagonals of the form (i, j) with $i < j$.

Draw an E step at the end.

Generalized triangulations

Definition. A j*-crossing* is a set of j diagonals where any two of them cross.



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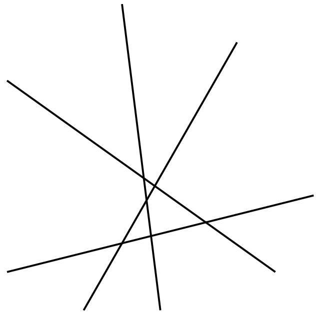

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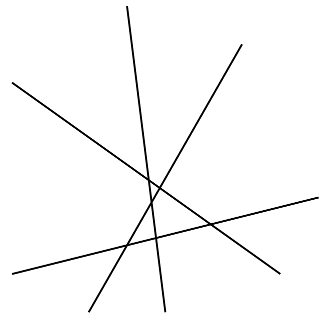
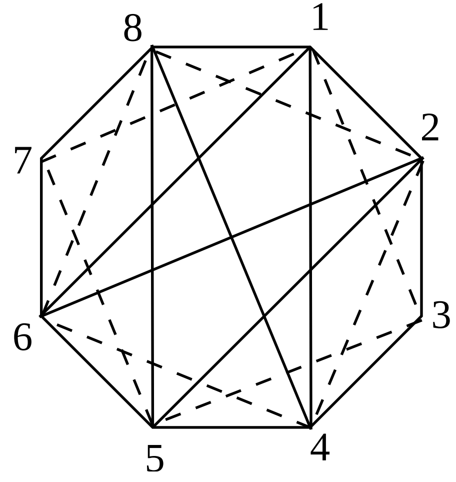


We could have defined a triangulation of $\mathfrak{C}_n$ as a maximal set of diagonals with no 2-crossings.

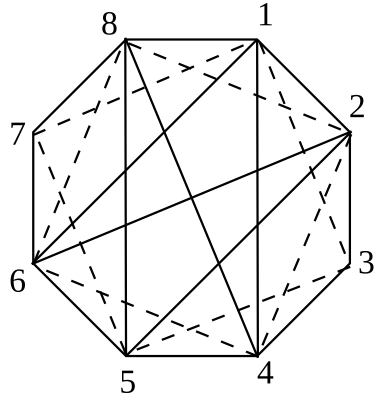
Definition. A k-triangulation of $\mathfrak{C}_n$ is a maximal set of diagonals with no $(k+1)$-crossings.



a 2-triangulation

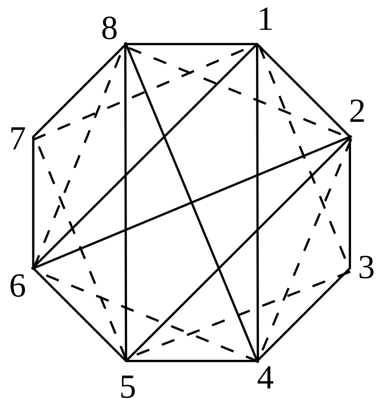
Facts about generalized triangulations

All the diagonals connecting vertices at distance $\leq k$ belong to every k-triangulation.



Facts about generalized triangulations

All the diagonals connecting vertices at distance $\leq k$ belong to every k-triangulation. We ignore such diagonals.



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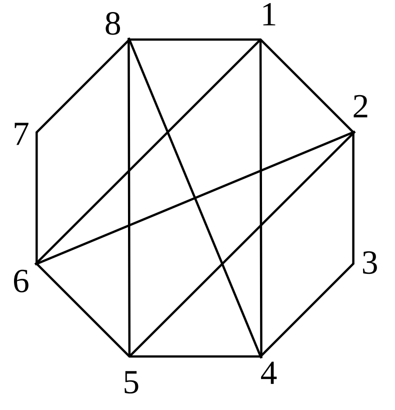

This 2-triangulation has $2(8 - 2 \cdot 2 - 1) = 6$ diagonals.

Theorem (Nakamigawa, Dress-Koolen-Moulton)**.**

Every k-triangulation of $\mathfrak{C}_n$ has exactly $k(n - 2k - 1)$ diagonals.

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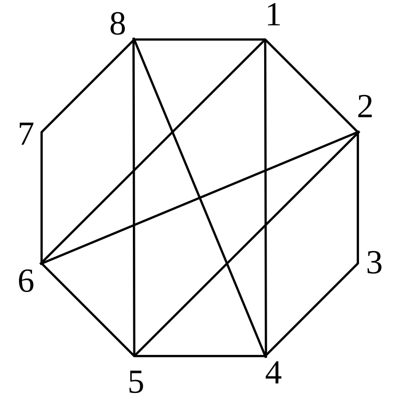


This 2-triangulation has $2(8-2\cdot 2-1) = 6$ diagonals.

Theorem (Nakamigawa, Dress-Koolen-Moulton)**.**

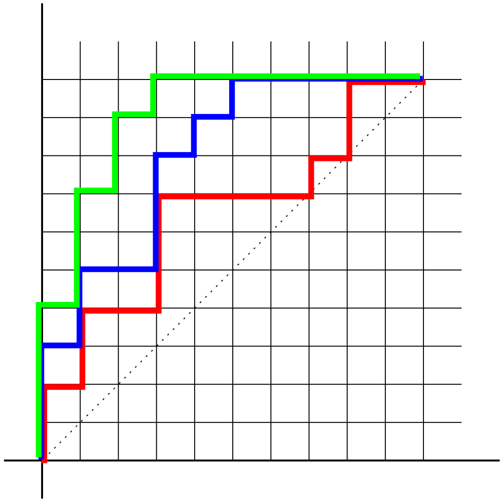
Every k-triangulation of $\mathfrak{C}_n$ has exactly $k(n-2k-1)$ diagonals.

Theorem (Jonsson)**.** *The number of k-triangulations of $\mathfrak{C}_n$ is*

$$\det(C_{n-i-j})_{i,j=1}^{k} = \begin{vmatrix} C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \end{vmatrix}.$$

Non-crossing Dyck paths

Theorem (Lindström, Gessel-Viennot)**.** *The number of k-tuples $(P_1, P_2, \ldots, P_k)$ of Dyck paths of size $n-2k$ such that each P_i never goes below P_{i+1} is given by the same determinant $\det(C_{n-i-j})_{i,j=1}^k$.*



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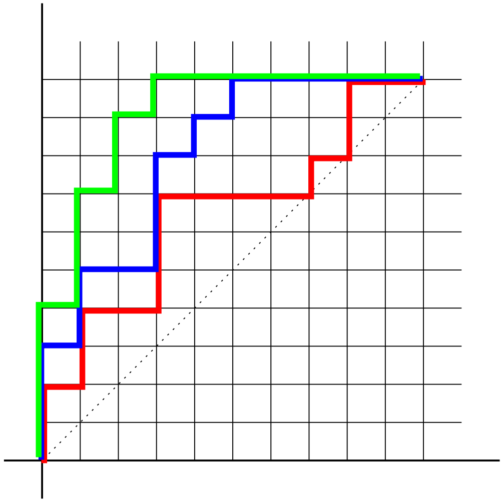

Problem: Find a bijection between k-triangulations and k-tuples of non-crossing Dyck paths.

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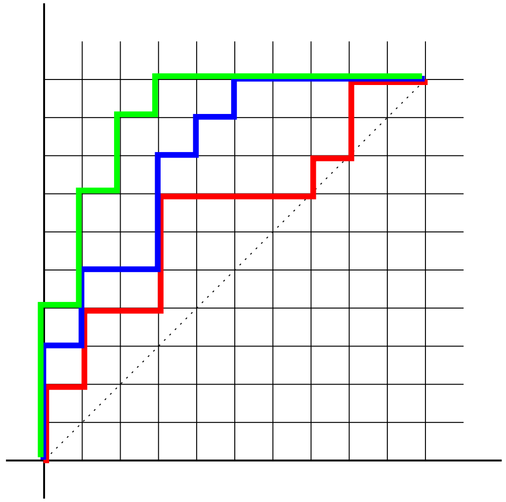
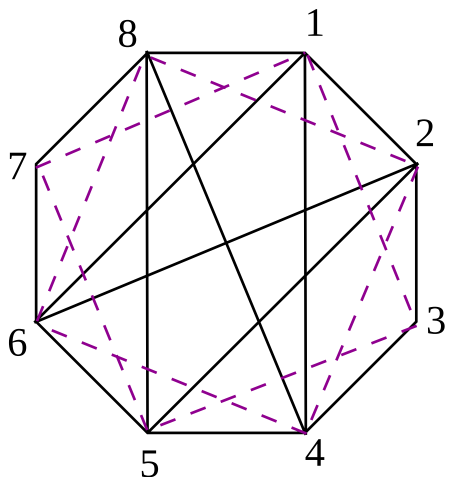
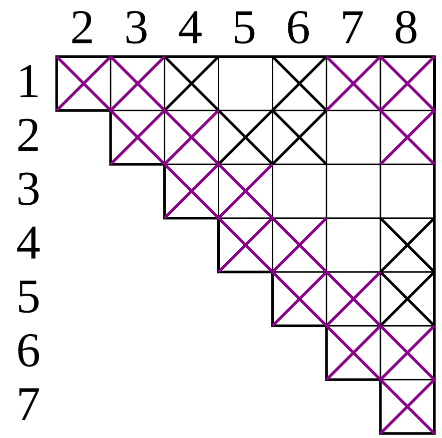


Problem: Find a bijection between k-triangulations and k-tuples of non-crossing Dyck paths.

- $k = 1 \longrightarrow$ known
- $k = 2 \longrightarrow$ we will see it next
- $k \geq 3 \longrightarrow$ open problem

We represent a 2-triangulation as an array:





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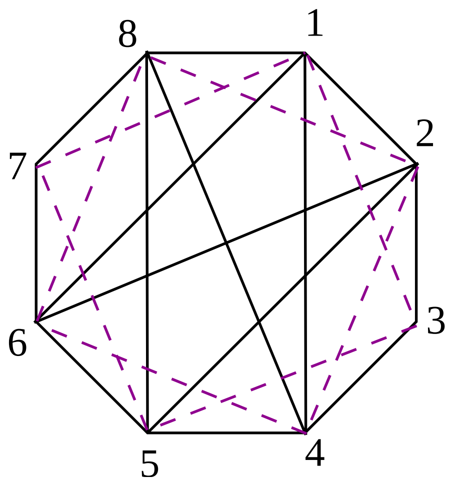
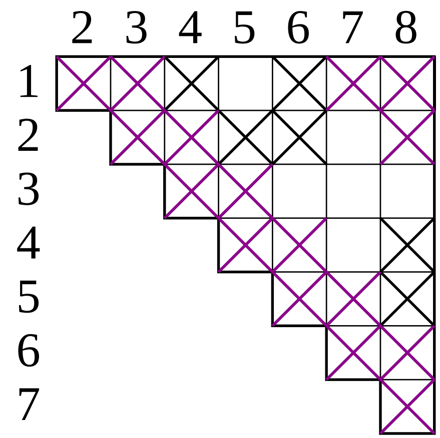
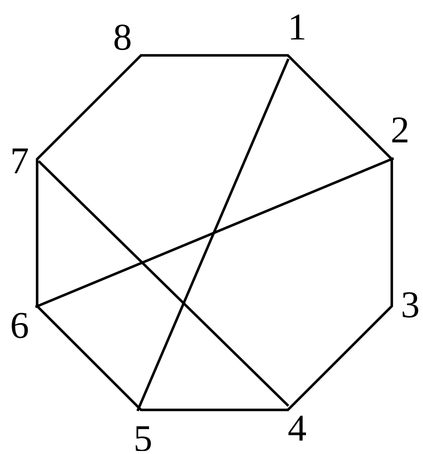
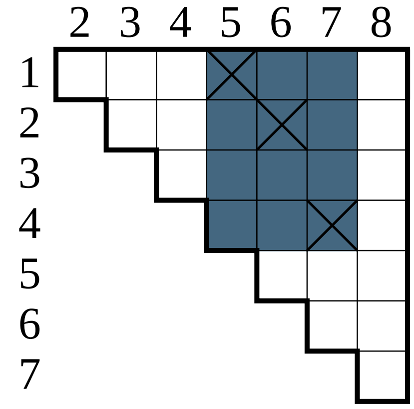




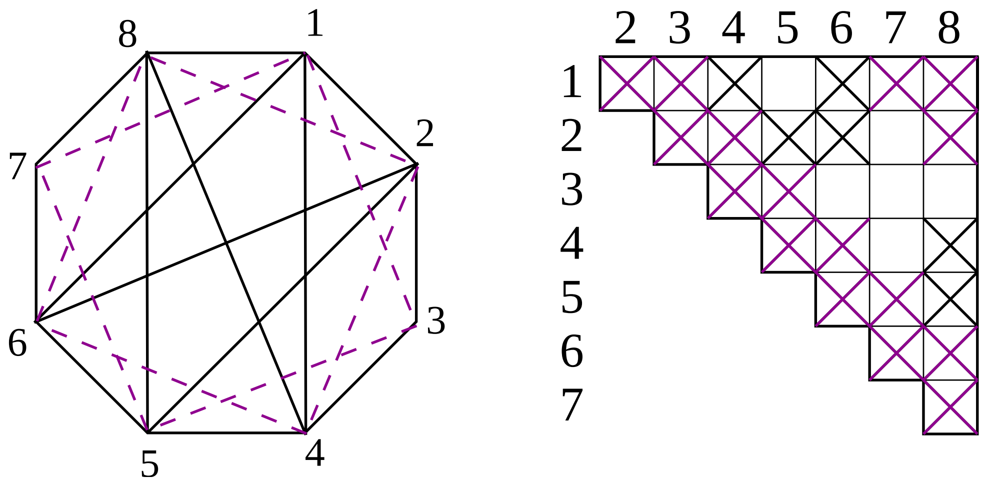
A 3-crossing would be





where the blue rectangle containing the crosses is inside the array.

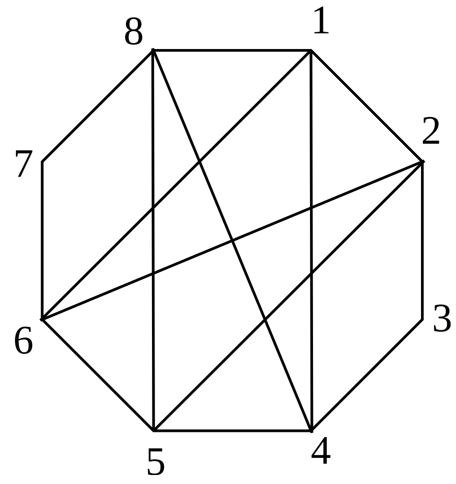
We represent a 2-triangulation as an array:

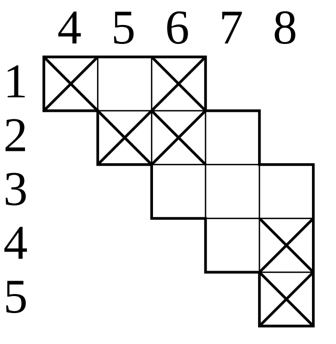


The purple crosses appear in any 2-triangulation, and they can't be part of any 3-crossing.

2-triangulations

We represent a 2-triangulation as an array:





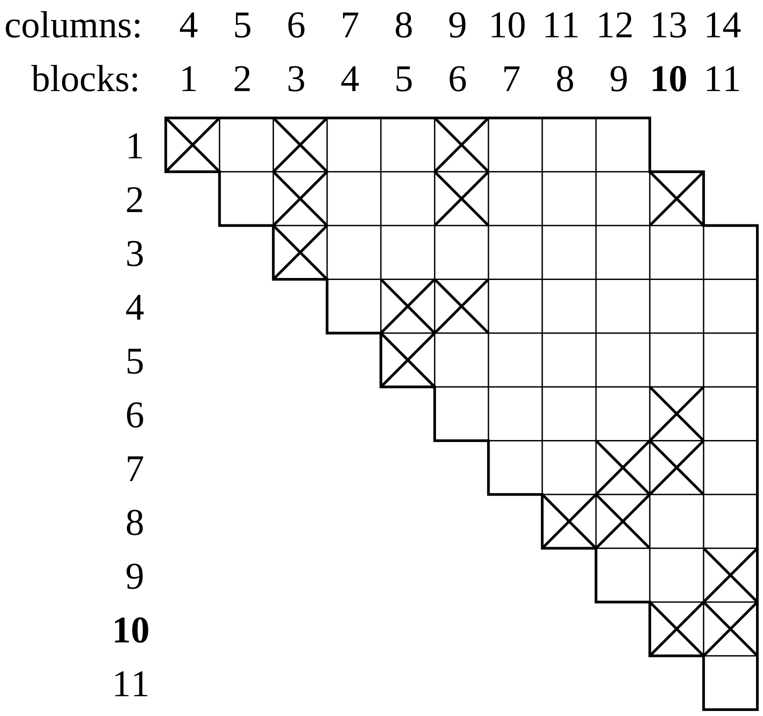
The purple crosses appear in any 2-triangulation, and they can't be part of any 3-crossing.

We omit them for simplicity.

Now there are exactly $2n - 10$ diagonals.

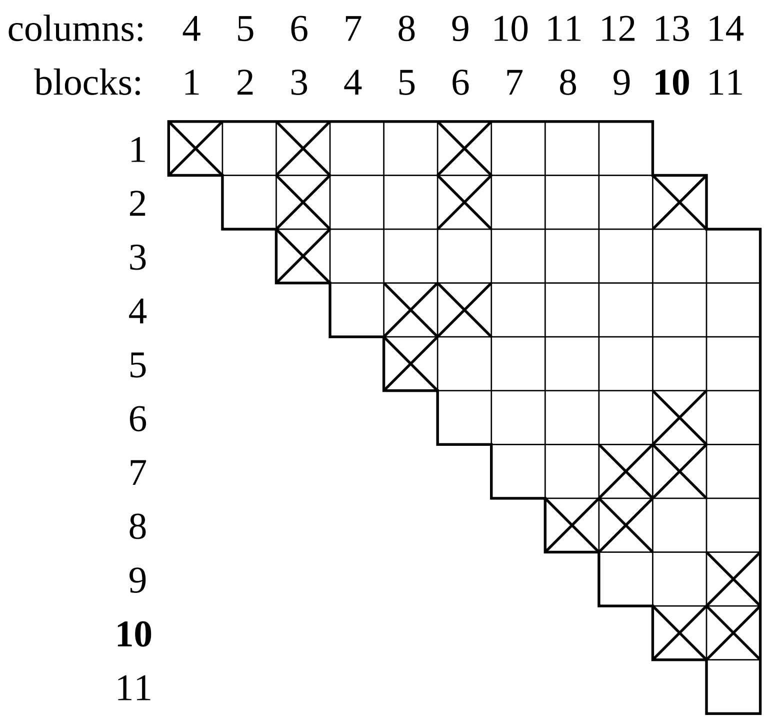
We will construct a bijection between 2-triangulations and pairs of non-crossing Dyck paths.

Given a 2-triangulation, first we give an algorithm to color half of the crosses blue and the other half red.



At each iteration, one cross will be colored red and another blue, and two blocks will be merged.

The bijection (part I): coloring stage



$r = 10$

Repeat until all crosses have been colored:

- Let r be the largest index so that row r has a cross in block r.
- Color blue the leftmost uncolored cross in block r.
- Merge blocks $r-2$ and $r-1$.
  (If $r = 2$, we consider that block 1 disappears when it is merged with "block 0".)
- Color red the rightmost uncolored cross in the merged block.

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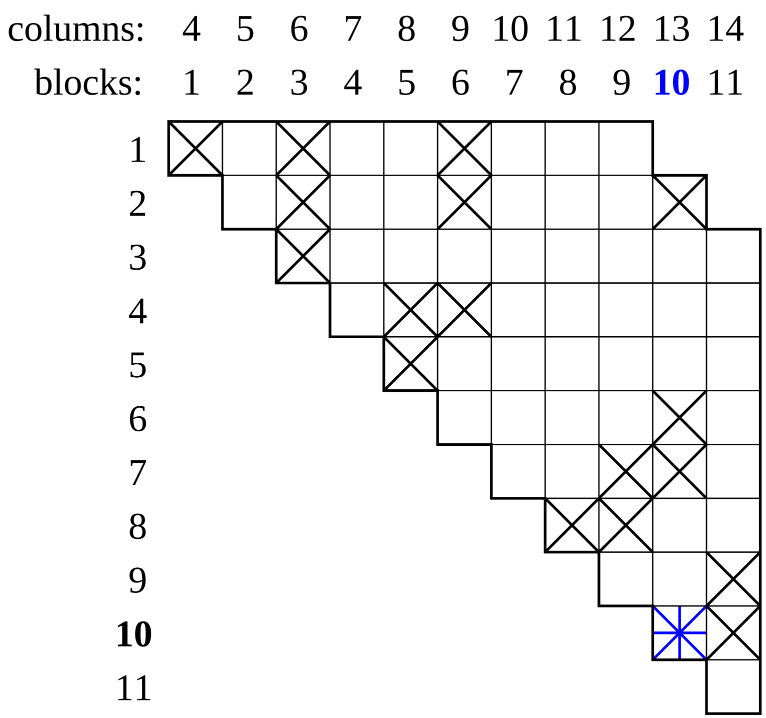

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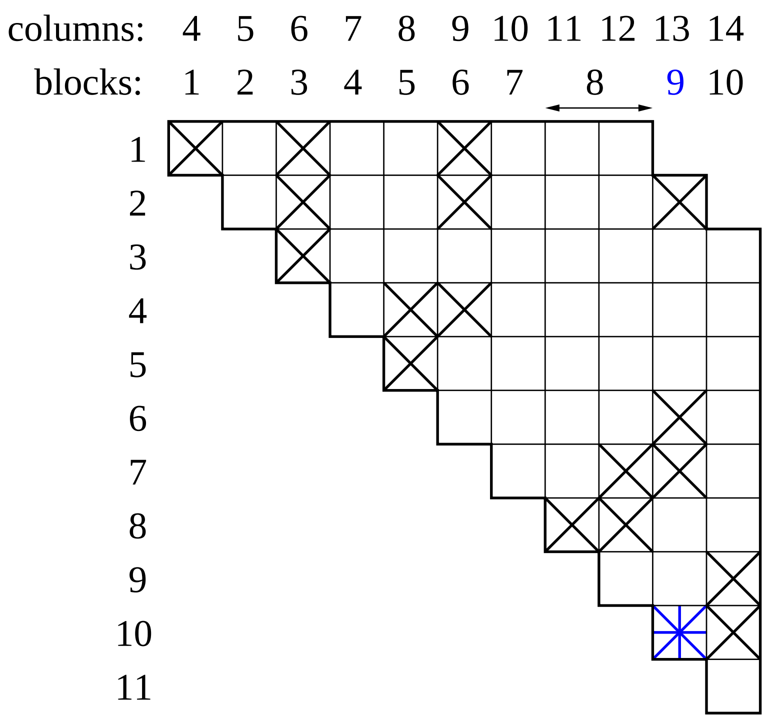

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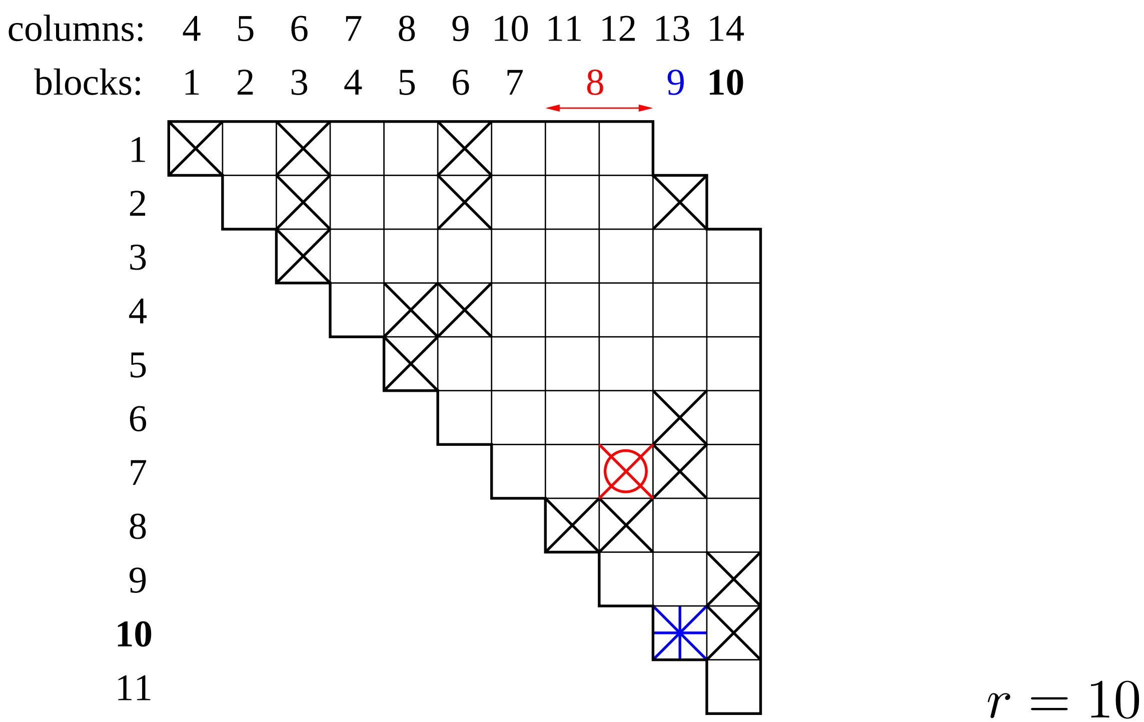

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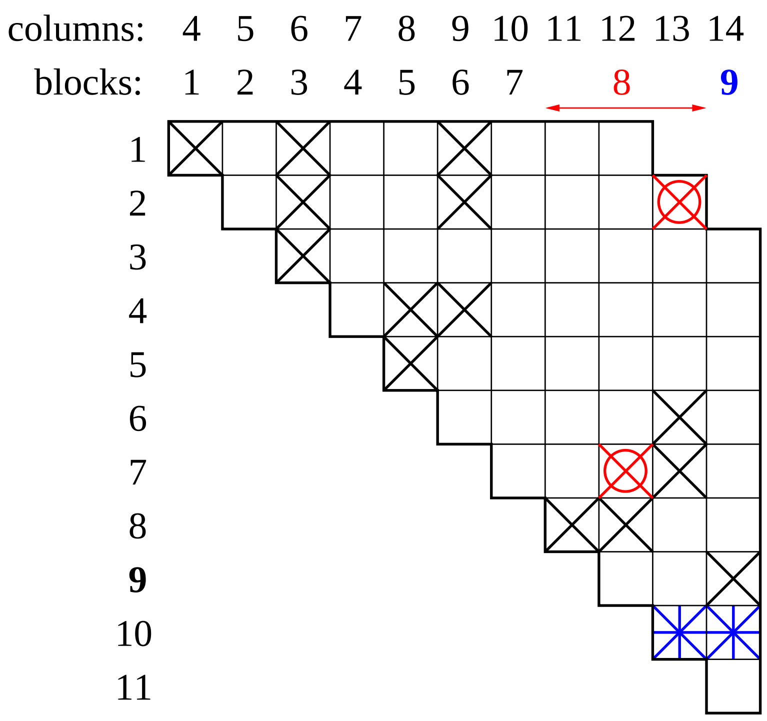

$r = 9$

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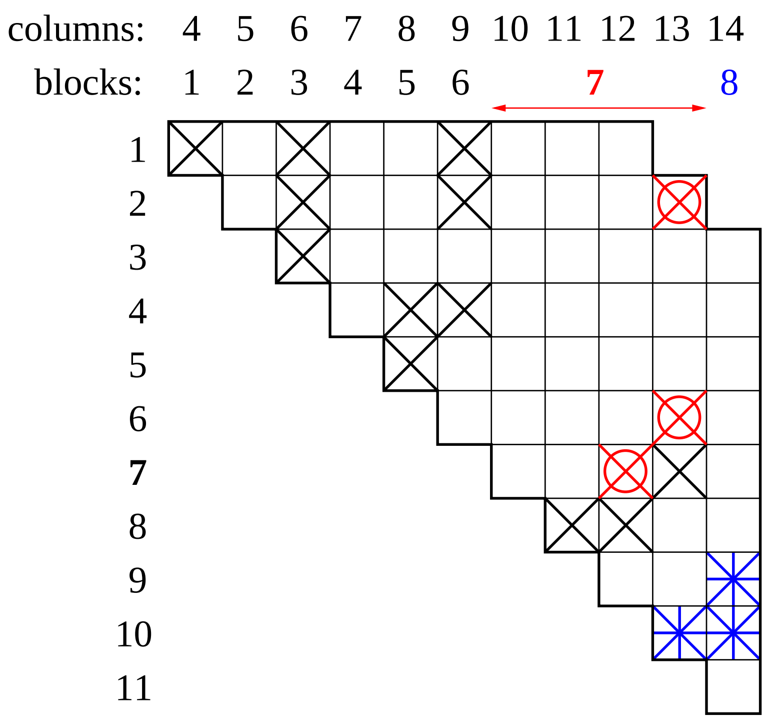

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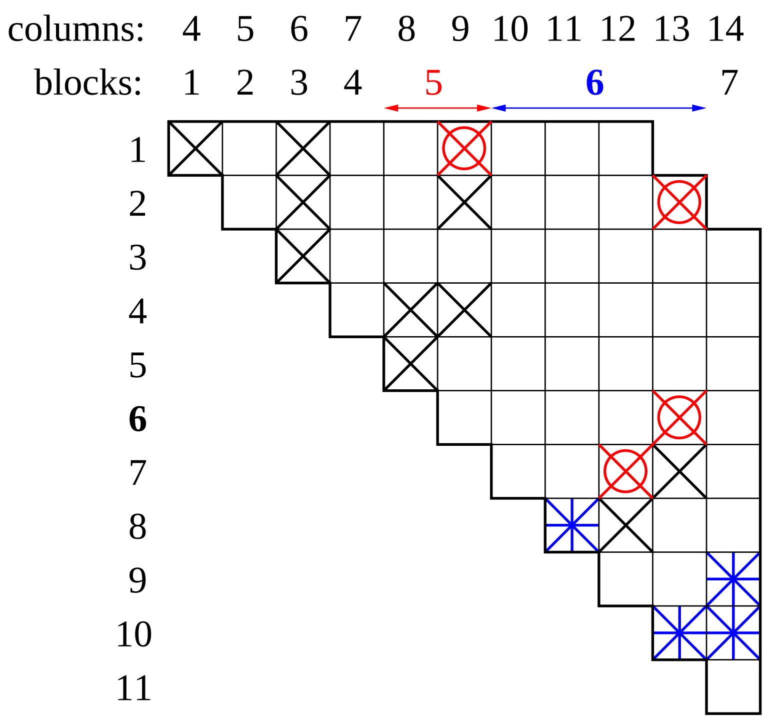

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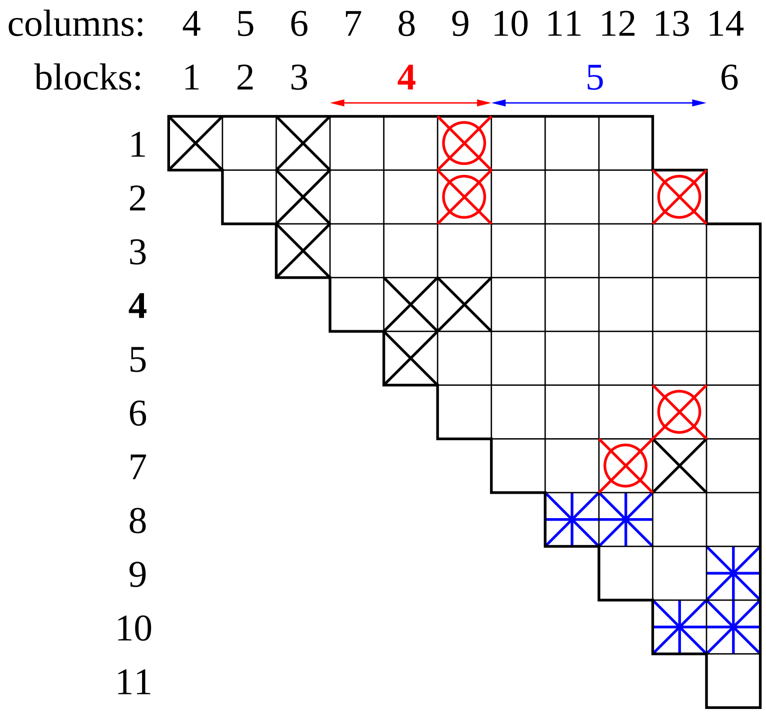

$r = 4$

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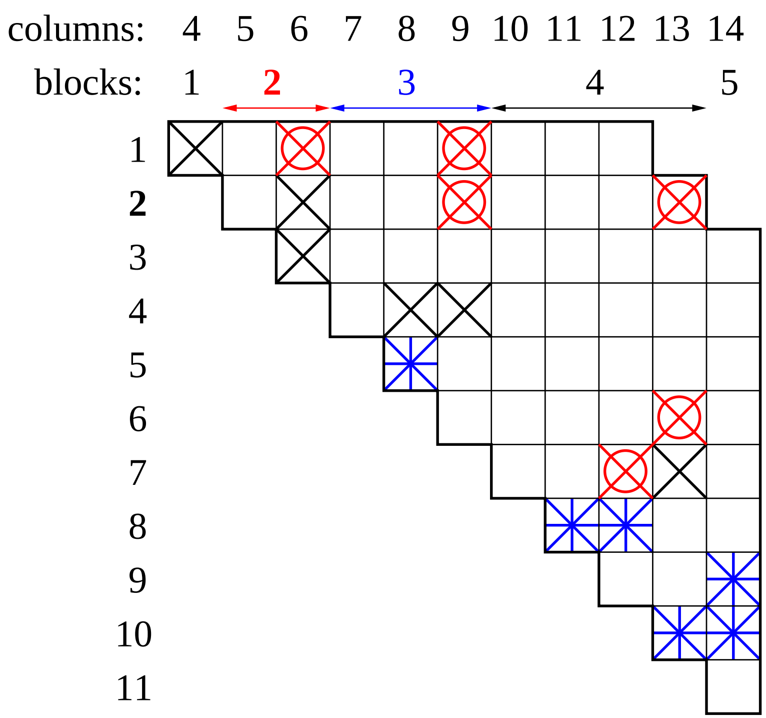

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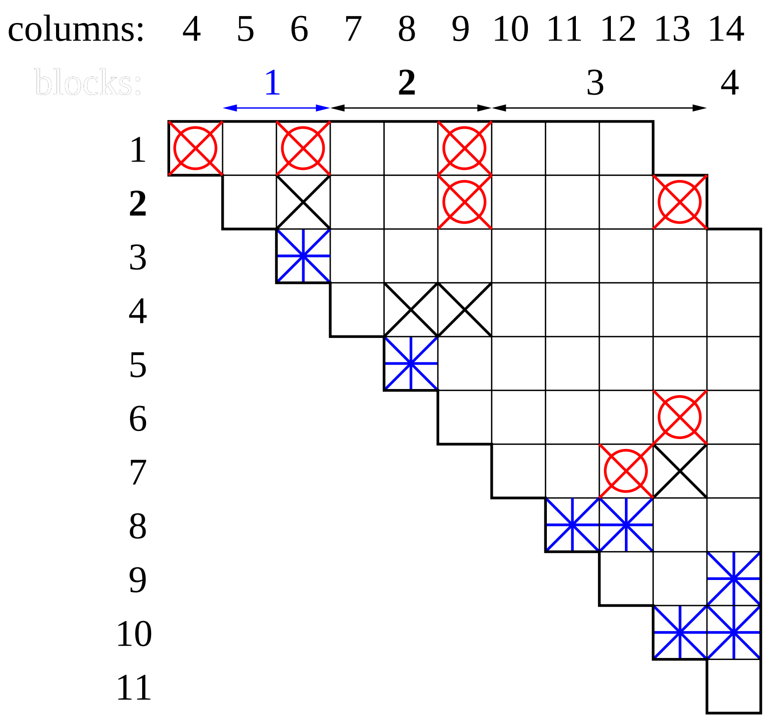

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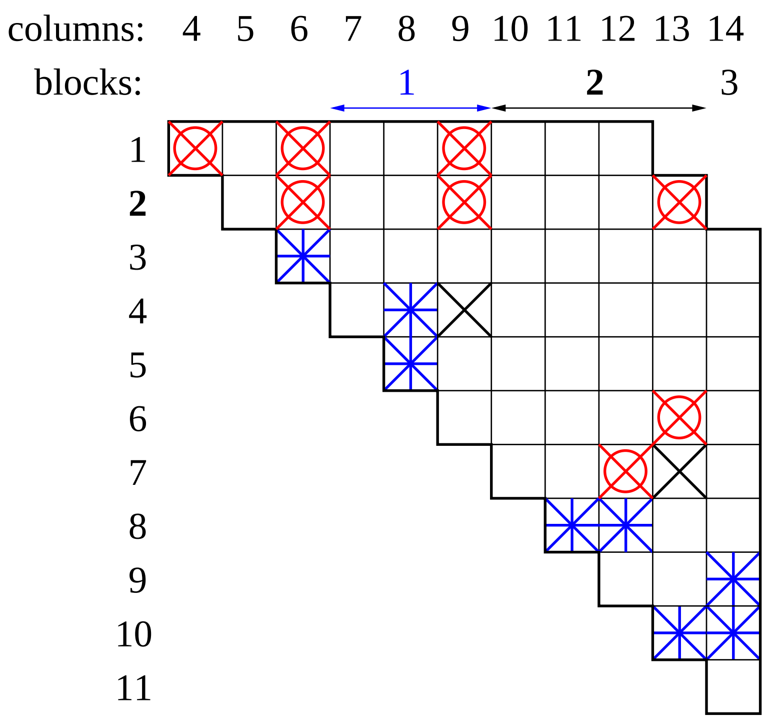

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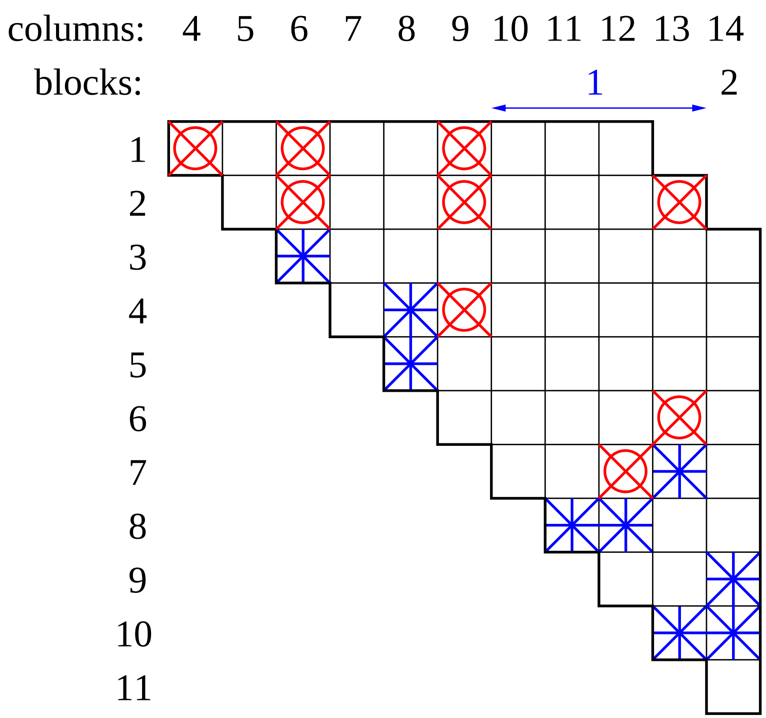

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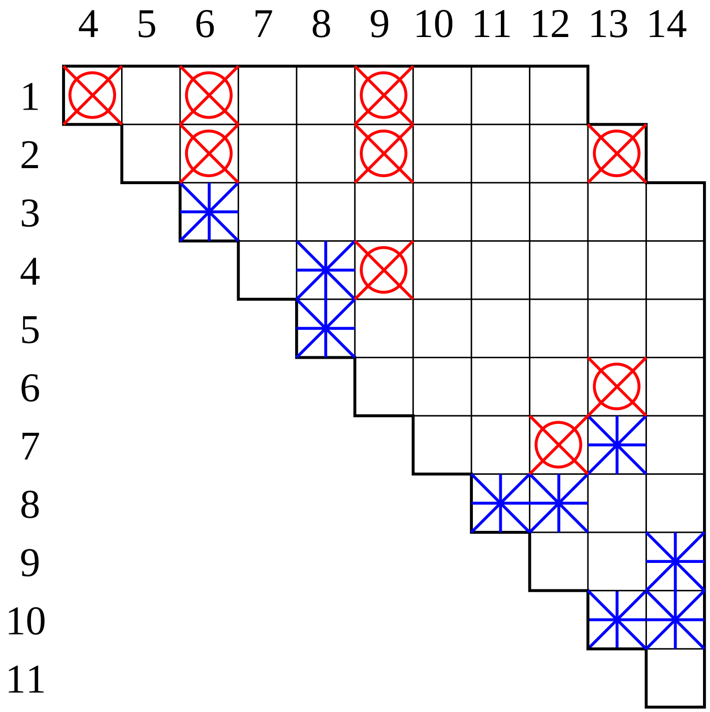
The bijection (part II): from colored crosses to paths



α_i := # blue crosses in column i

β_i := # red crosses in column i

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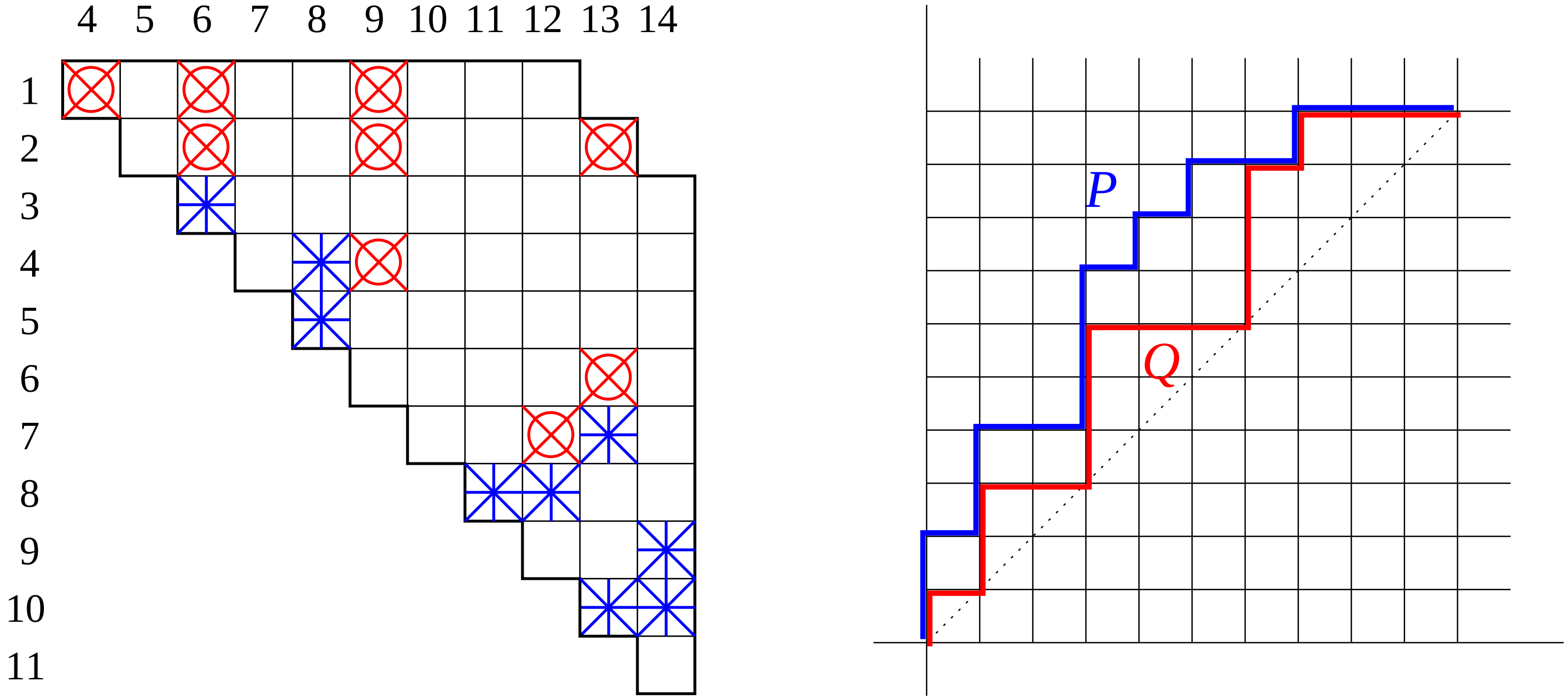


$\alpha_i :=$ # blue crosses in column i
$\beta_i :=$ # red crosses in column i

Define

$$P = NE^{\alpha_5} NE^{\alpha_6} \cdots NE^{\alpha_{n-1}} NE^{\alpha_n} E$$
$$Q = NE^{\beta_4} NE^{\beta_5} \cdots NE^{\beta_{n-2}} NE^{\beta_{n-1}} E$$

Idea of the proof:

- Construct a generating tree for 2-triangulations.
- Construct a generating tree for pairs of non-crossing Dyck paths.
- Give an isomorphism between the generating trees.

Why does the bijection work?

Idea of the proof:

- Construct a generating tree for 2-triangulations.
- Construct a generating tree for pairs of non-crossing Dyck paths.
- Give an isomorphism between the generating trees.

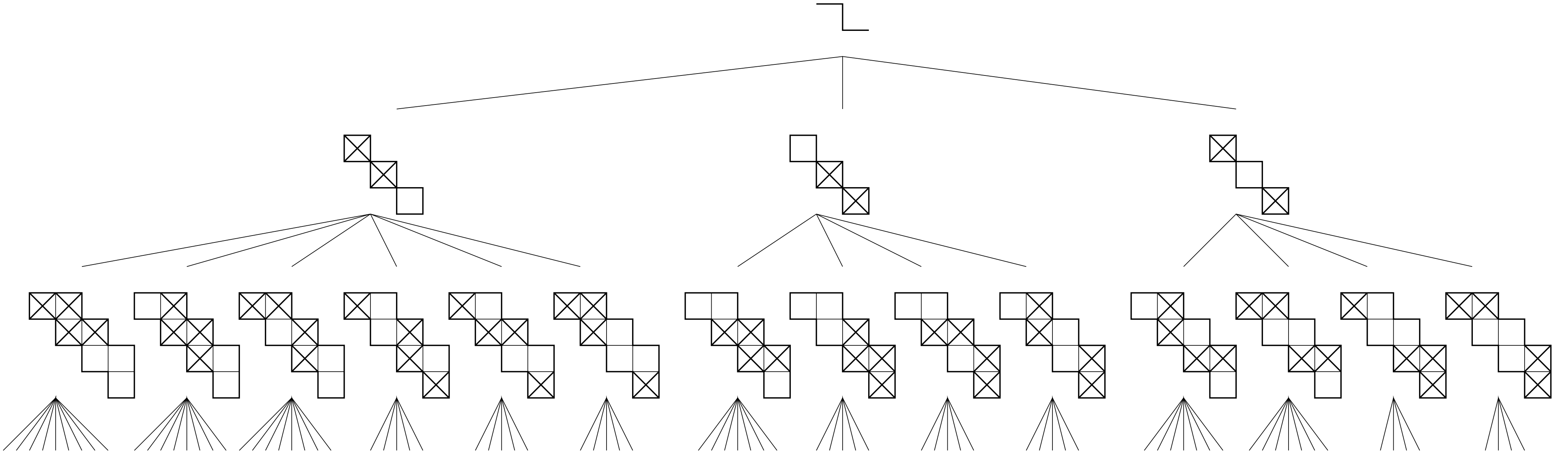
Each node of a *generating tree* represents one of our objects.

Nodes at the same level represent objects of the same size.

A node at level ℓ has its children at level $\ell + 1$.

There is a rule that describes the children of any given node.

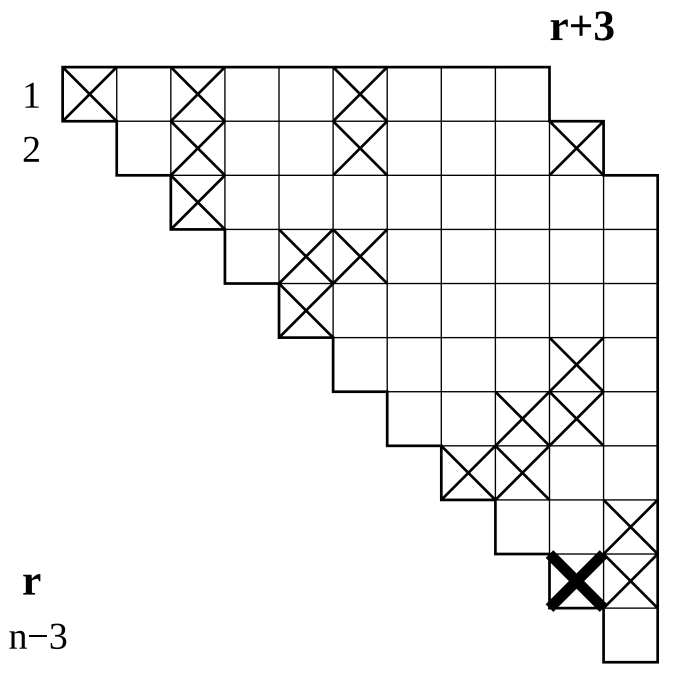
A generating tree for 2-triangulations



The nodes at level ℓ represent the 2-triangulations of an $(\ell + 5)$-gon.

The children of a 2-triangulation

To generate a child of a 2-triangulation of an n-gon:



- Let r be the largest so that there is a diagonal $(r, r+3)$.

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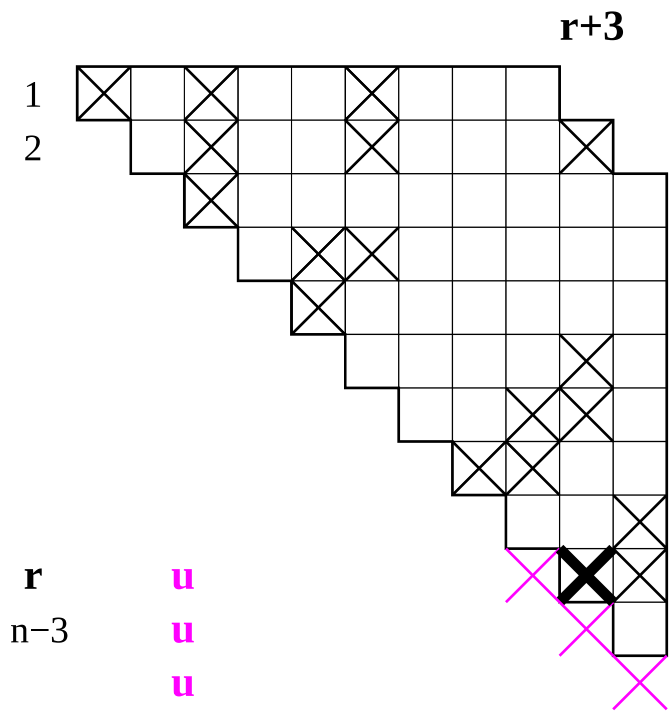

- Let r be the largest so that there is a diagonal $(r, r+3)$.
- Choose $u \in \{r, \ldots, n-2\}$. Add a cross in position $(u, u+2)$.

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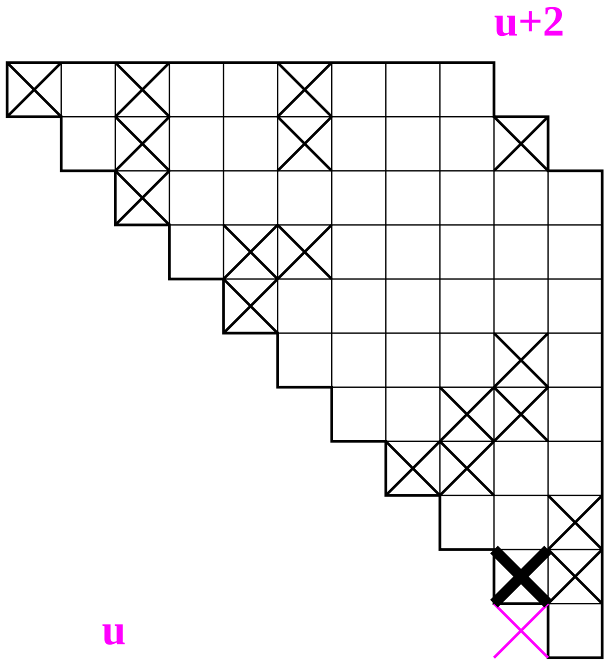

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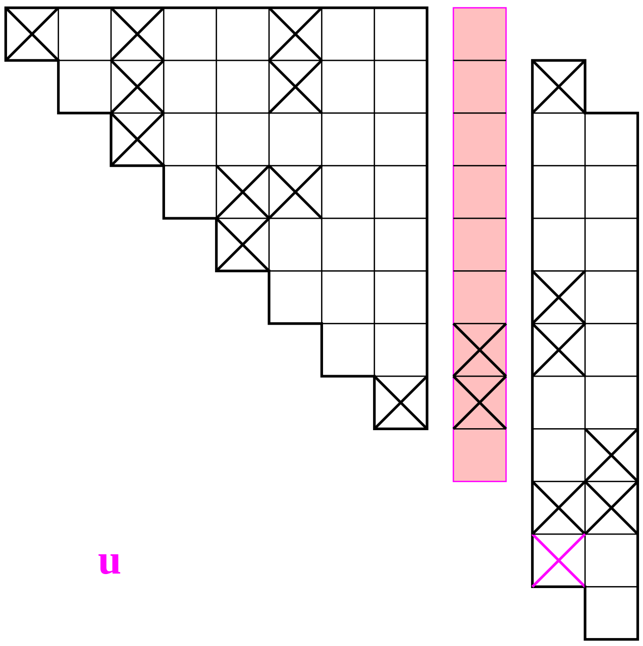

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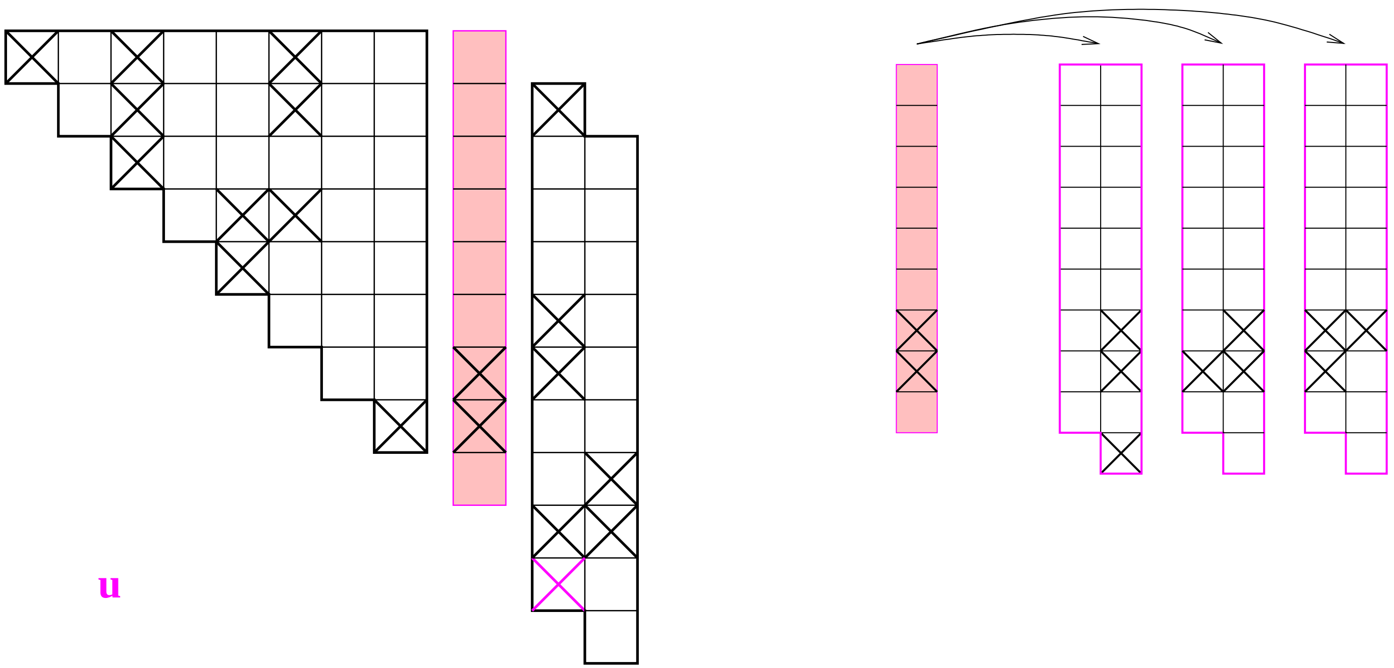

- Choose $u \in \{r, \ldots, n-2\}$. Add a cross in position $(u, u+2)$.
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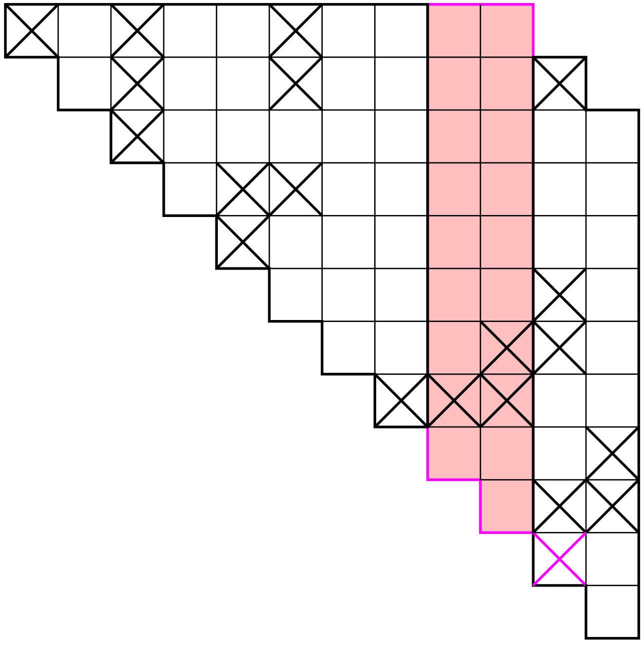

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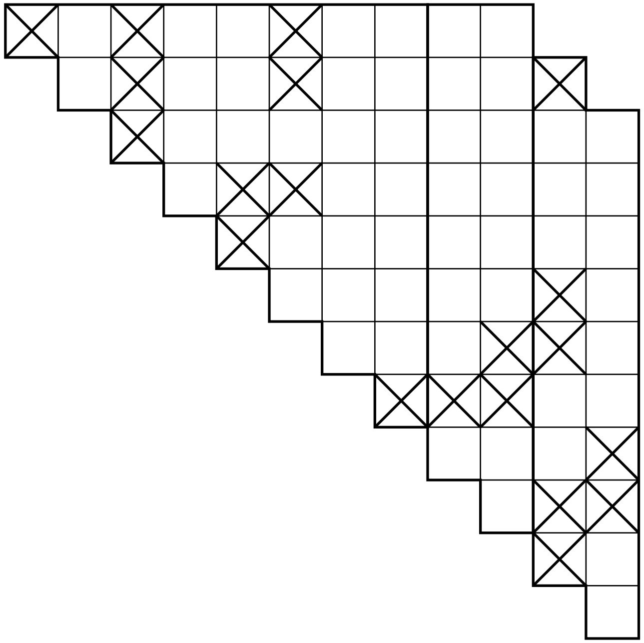

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Labeling the nodes

Given a 2-triangulation T, let

- $r = \max\{a \ : \ (a, a+3) \in T\}$,
- $h_j =$ number of crosses in column j of the diagram of T.

We associate the label $(h_{r+1}, \ldots, h_{n-1})$ to it.

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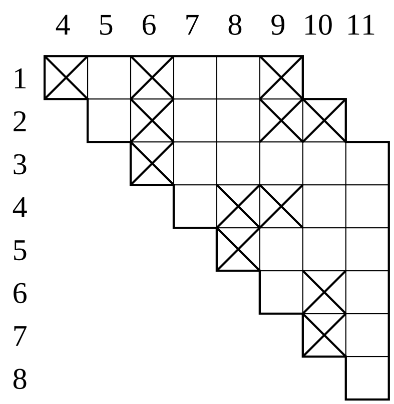

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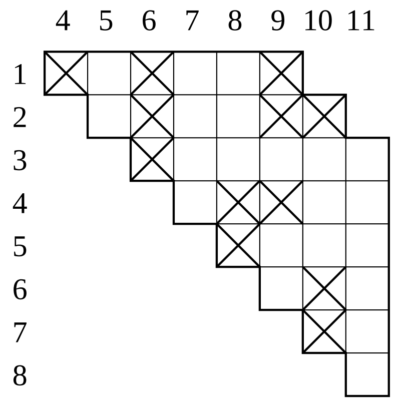

has label $(2, 3, 3)$.

This label determines the labels of its children, according to the generating rule

$$(d_1, d_2, \ldots, d_s) \longrightarrow \begin{array}{l} \{(i, d_j - i + 1, d_{j+1} + 1, d_{j+2}, \ldots, d_s) \ : \ 1 \le j \le s-1, \ 0 \le i \le d_j\} \\ \cup \, \{(i, d_s - i + 1) \ : \ 0 \le i \le d_s + 1\}. \end{array}$$

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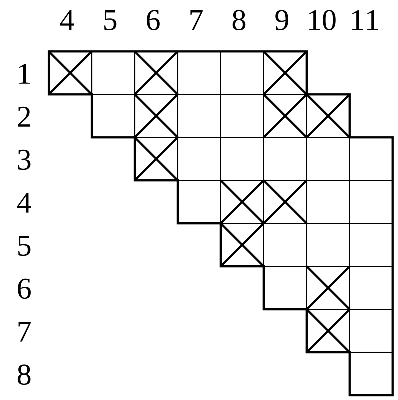

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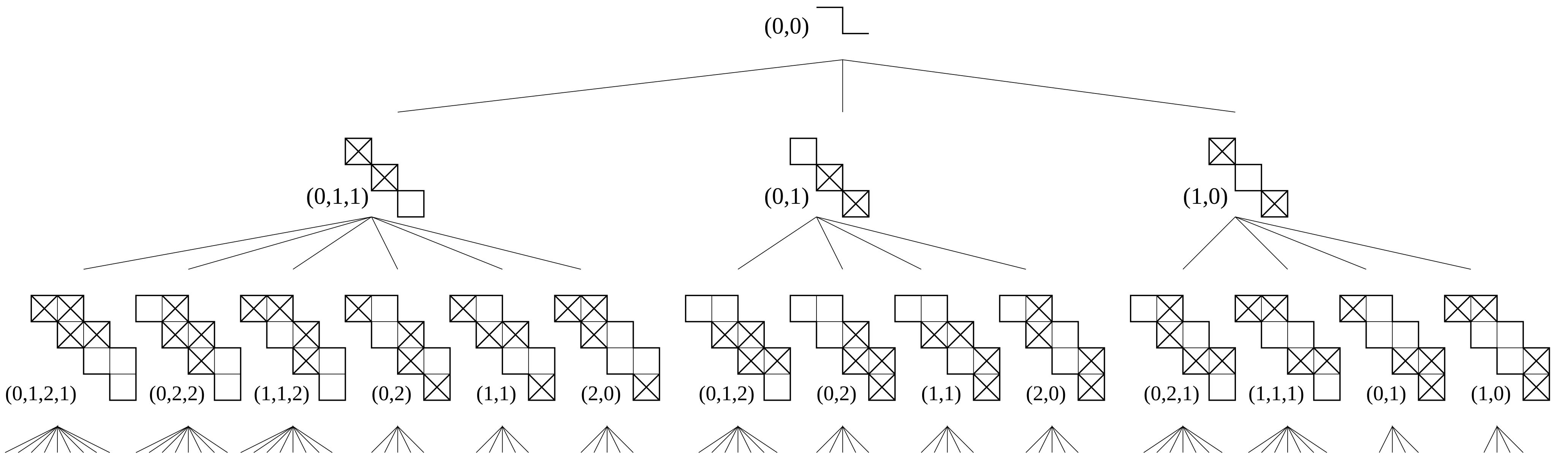
$$(0,1,3,2) \longrightarrow (0,1,2,3,2),\ (0,2,4,2),\ (1,1,4,2),\ (0,4,3),\ (1,3,3),\ (2,2,3),\ (3,1,3),$$
$$(0,3),\ (1,2),\ (2,1),\ (3,0).$$

The generating tree with labels

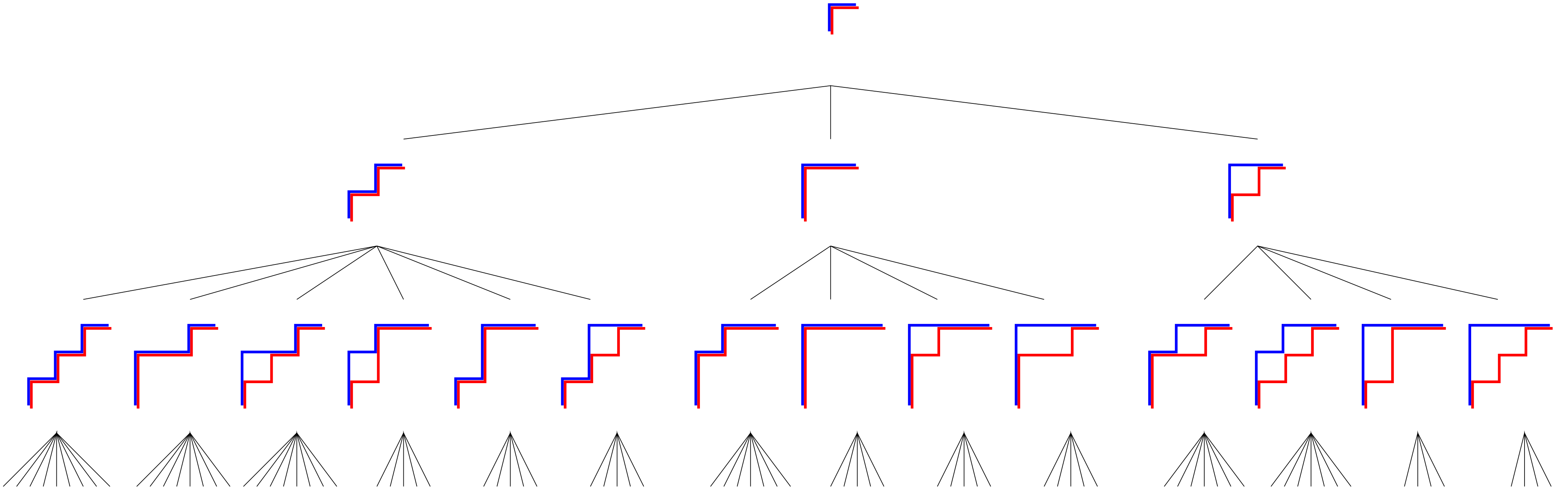


Root: $(0,0)$

Generating rule:

$$(d_1, d_2, \ldots, d_s) \longrightarrow \begin{array}{l} \{(i, d_j - i + 1, d_{j+1} + 1, d_{j+2}, \ldots, d_s) \;:\; 1 \le j \le s-1,\ 0 \le i \le d_j\} \\ \cup \{(i, d_s - i + 1) \;:\; 0 \le i \le d_s + 1\}. \end{array}$$

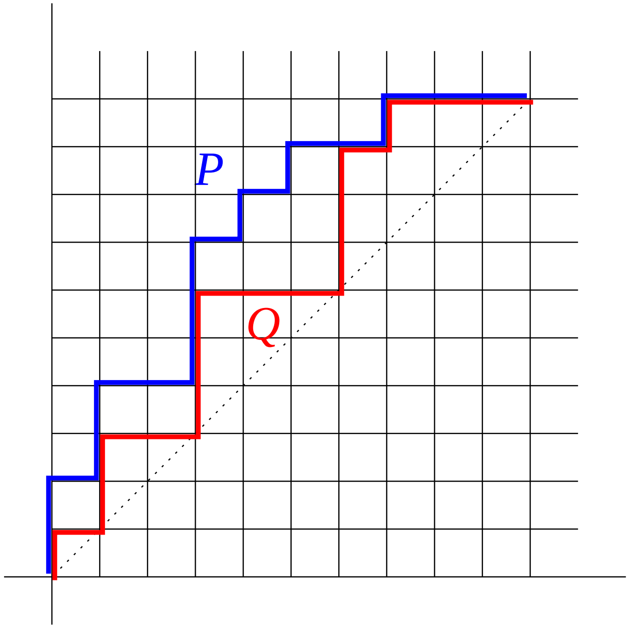
A generating tree for pairs of non-crossing Dyck paths

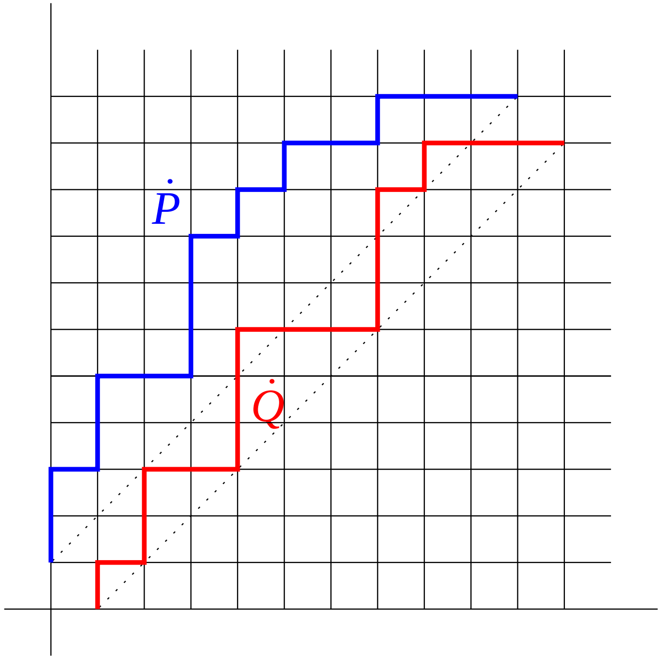


The nodes at level ℓ represent pairs of paths of size $\ell + 1$.

Non-crossing Dyck paths

We will draw the upper path starting at $(0, 1)$ and the lower path starting at $(1, 0)$.





The old paths being non-crossing is equivalent to the new paths being non-intersecting.

Encoding of pairs of Dyck paths

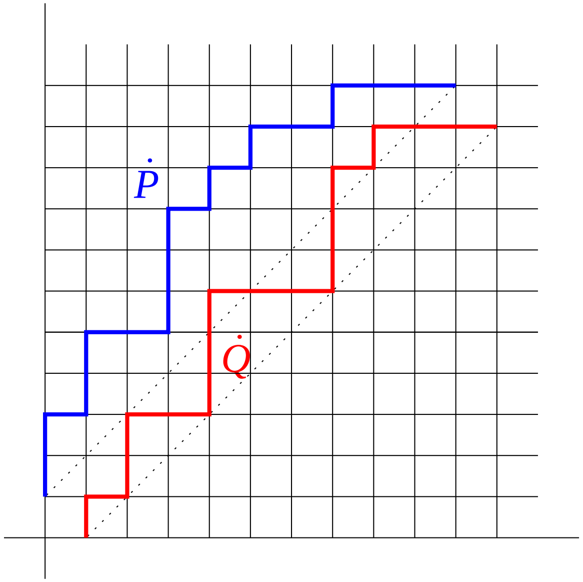
If

$$P = NE^{p_m} NE^{p_{m-1}} \cdots NE^{p_2} NE^{p_1} E$$
$$Q = NE^{q_m} NE^{q_{m-1}} \cdots NE^{q_2} NE^{q_1} E,$$

we encode the pair as

$$[P, Q] := \begin{bmatrix} p_{m+2} & p_{m+1} & p_m & p_{m-1} & \cdots & p_3 & p_2 & p_1 \\ q_{m+1} & q_m & q_{m-1} & q_{m-2} & \cdots & q_2 & q_1 & 0 \end{bmatrix},$$

where $p_{m+1} = q_{m+1} = p_{m+2} = 0$.



Example:

$$[P, Q] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & \mathbf{1} & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & \mathbf{0} & 1 & 2 & 0 \end{bmatrix}$$

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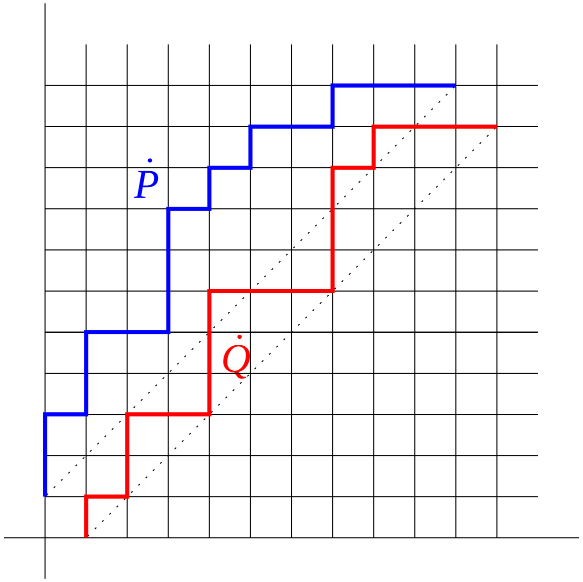
Let $s = s(P,Q) = \min\{j \geq 2 : p_j q_j = 0\}$.



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$$s(P,Q) = 3$$

The children of a pair of Dyck paths

Let

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Let $s = \min\{j \geq 2 : p_j q_j = 0\}$ as before.

Choose $t \in \{1, 2, \ldots, s\}$.

The children of a pair of Dyck paths

Let

$$[P,Q] := \left[\begin{array}{cccccccc} p_{m+2} & p_{m+1} & p_m & p_{m-1} & \cdots & p_3 & p_2 & p_1 \\ q_{m+1} & q_m & q_{m-1} & q_{m-2} & \cdots & q_2 & q_1 & 0 \end{array}\right].$$

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The following are encodings of children of (P, Q):

$$\left[\begin{array}{cccccccccc} p_{m+2} & p_{m+1} & \cdots & p_{t+2} & p_{t+1} - i & i & p_t + 1 & p_{t-1} & \cdots & p_2 & p_1 \\ q_{m+1} & q_m & \cdots & q_{t+1} & 0 & q_t + 1 & q_{t-1} & q_{t-2} & \cdots & q_1 & 0 \end{array}\right] \text{ for each } i \in \{1, \ldots, p_{t+1}\},$$

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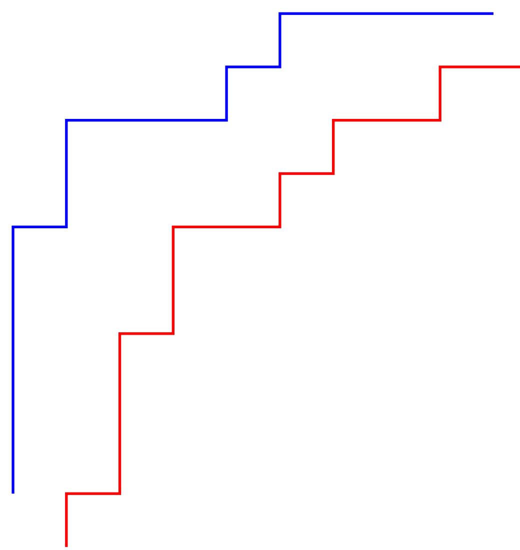
$$\left[\begin{array}{cccccccccc} p_{m+2} & p_{m+1} & \cdots & p_{t+2} & p_{t+1} & 0 & p_t + 1 & p_{t-1} & \cdots & p_2 & p_1 \\ q_{m+1} & q_m & \cdots & q_{t+1} & j & q_t - j + 1 & q_{t-1} & q_{t-2} & \cdots & q_1 & 0 \end{array}\right] \text{ for each } j \in \{1, \ldots, q_t + \delta_{1t}\}.$$

The children of a pair of Dyck paths: example

To generate a child of a pair of non-crossing Dyck paths of size m:



$$[P,Q] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \overbrace{\mathbf{0}}^{p_s} & 3 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & \underbrace{\mathbf{2}}_{q_s} & 1 & 2 & 1 & 0 \end{bmatrix}$$

$$s = 4$$

- $\longrightarrow \quad s = \min\{j \geq 2 : p_j q_j = 0\}$.
- Choose $t \in \{1, 2, \ldots, s\}$.
- Add an east step to Q at level t in and to P at level $t-1$.
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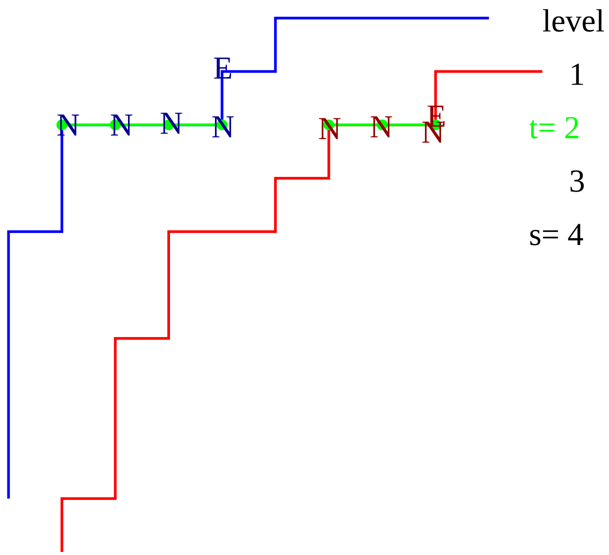

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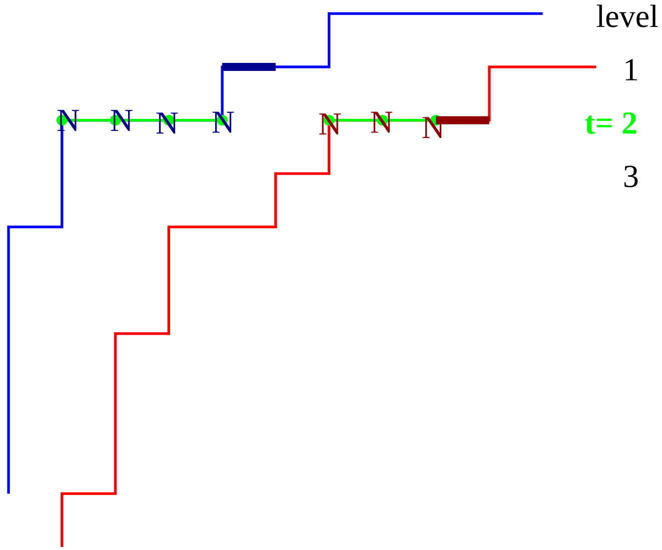

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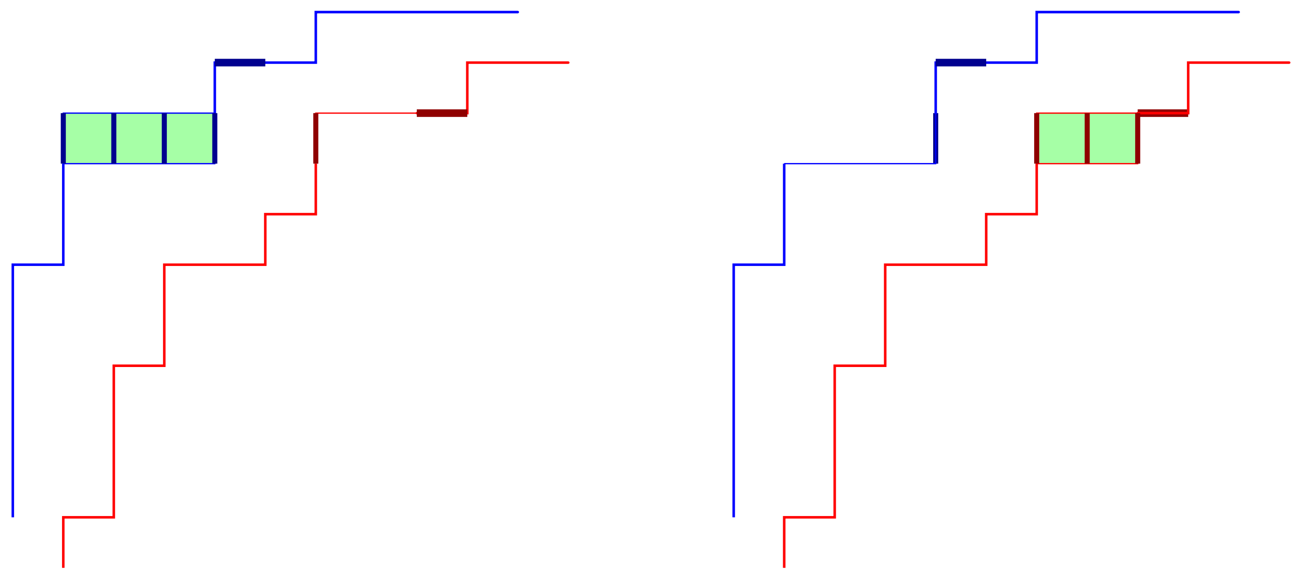

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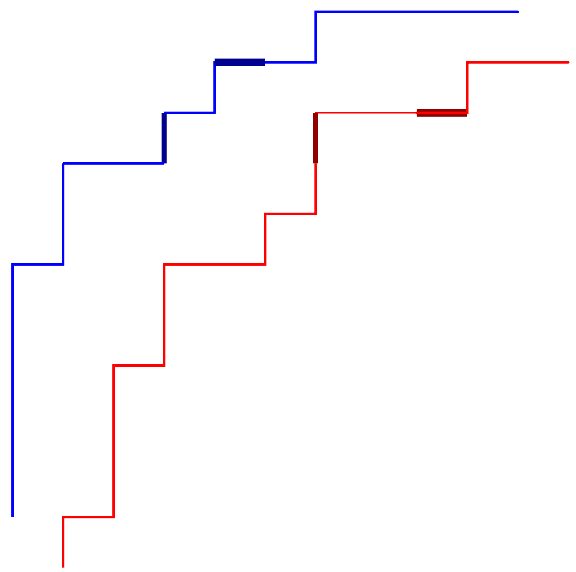

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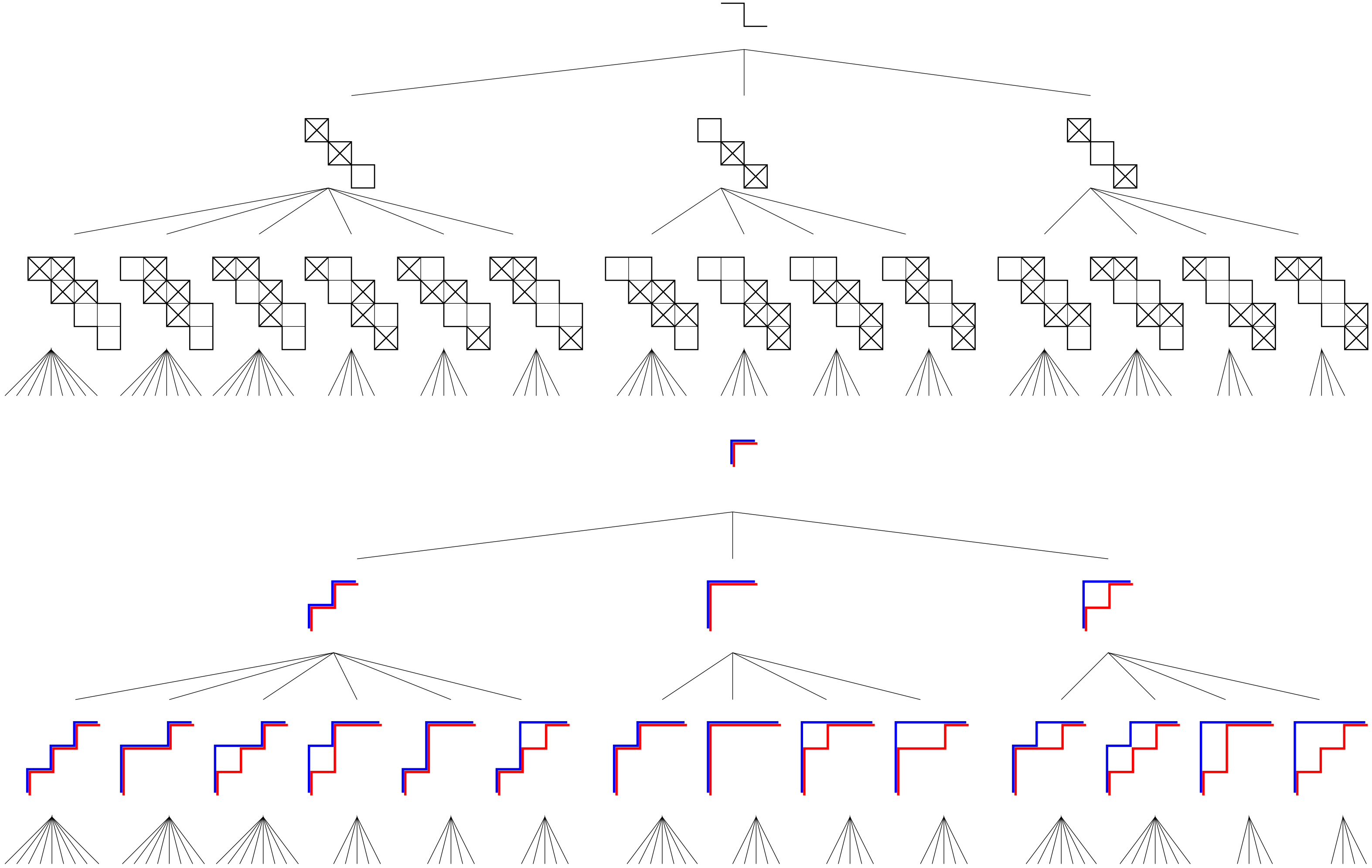
The two generating trees are isomorphic

These generating rules for 2-triangulations and for pairs of Dyck paths yield isomorphic generating trees.



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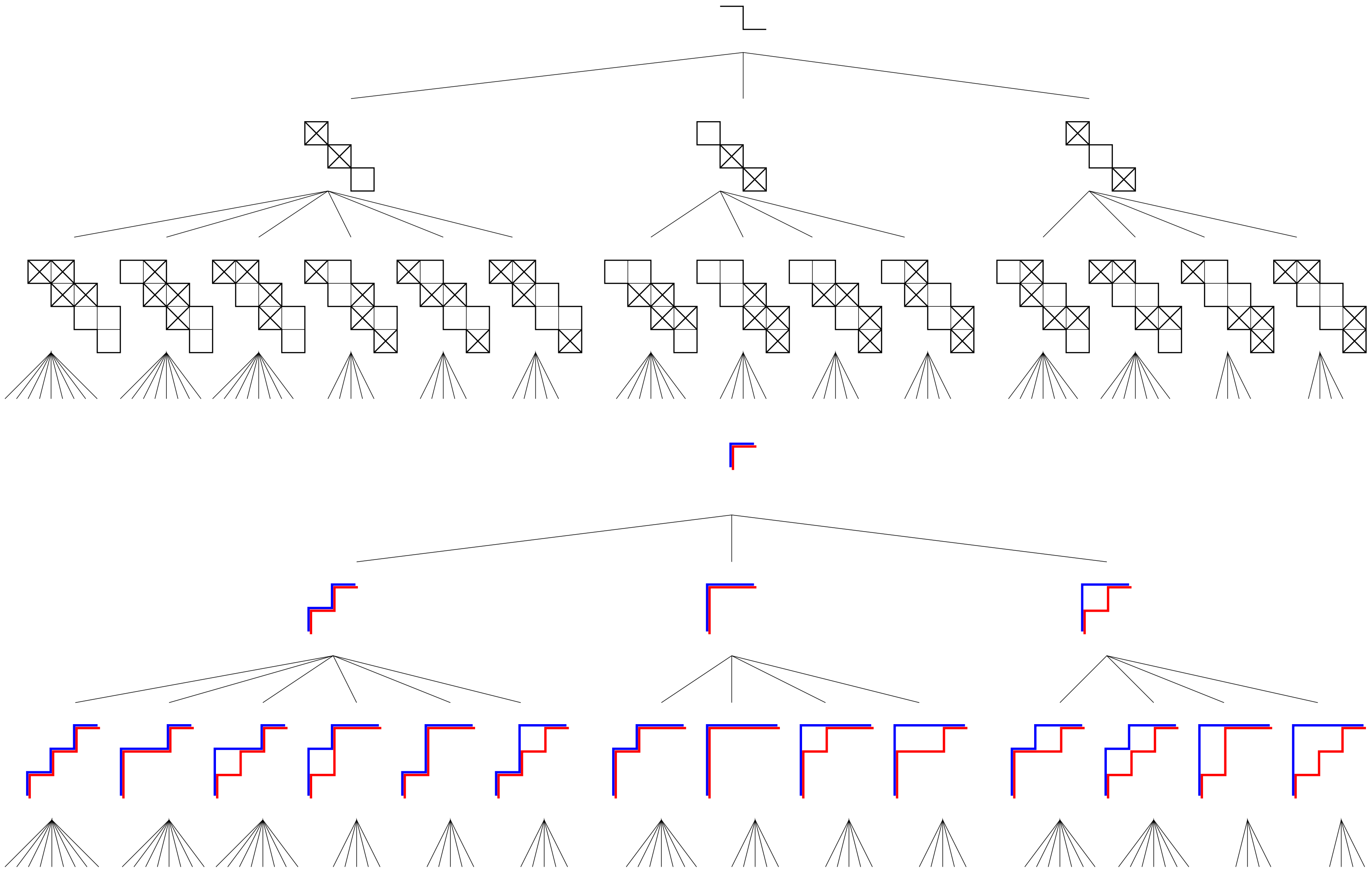
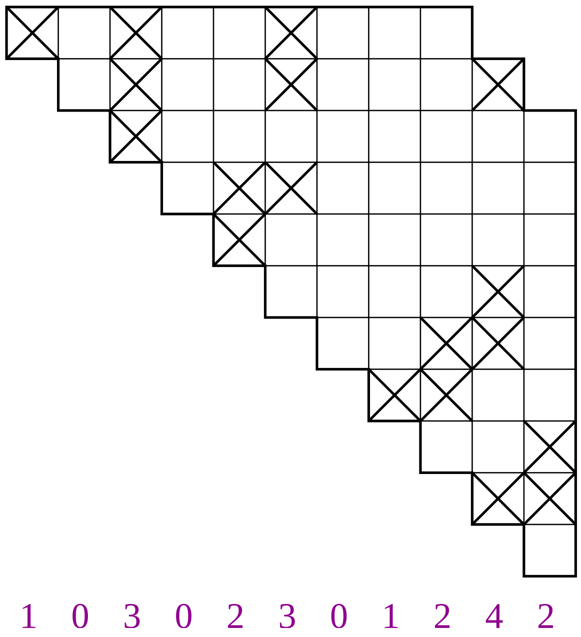
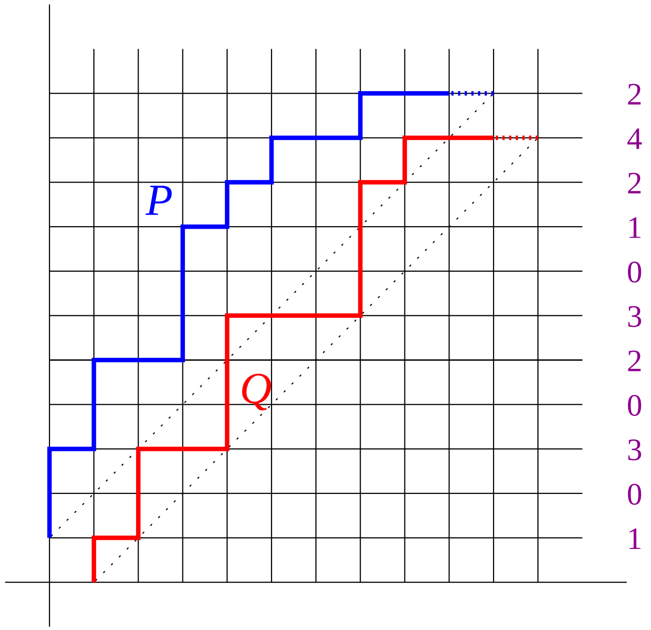


The bijection we described is just the one induced by the isomorphism of generating trees.

Some properties of the bijection

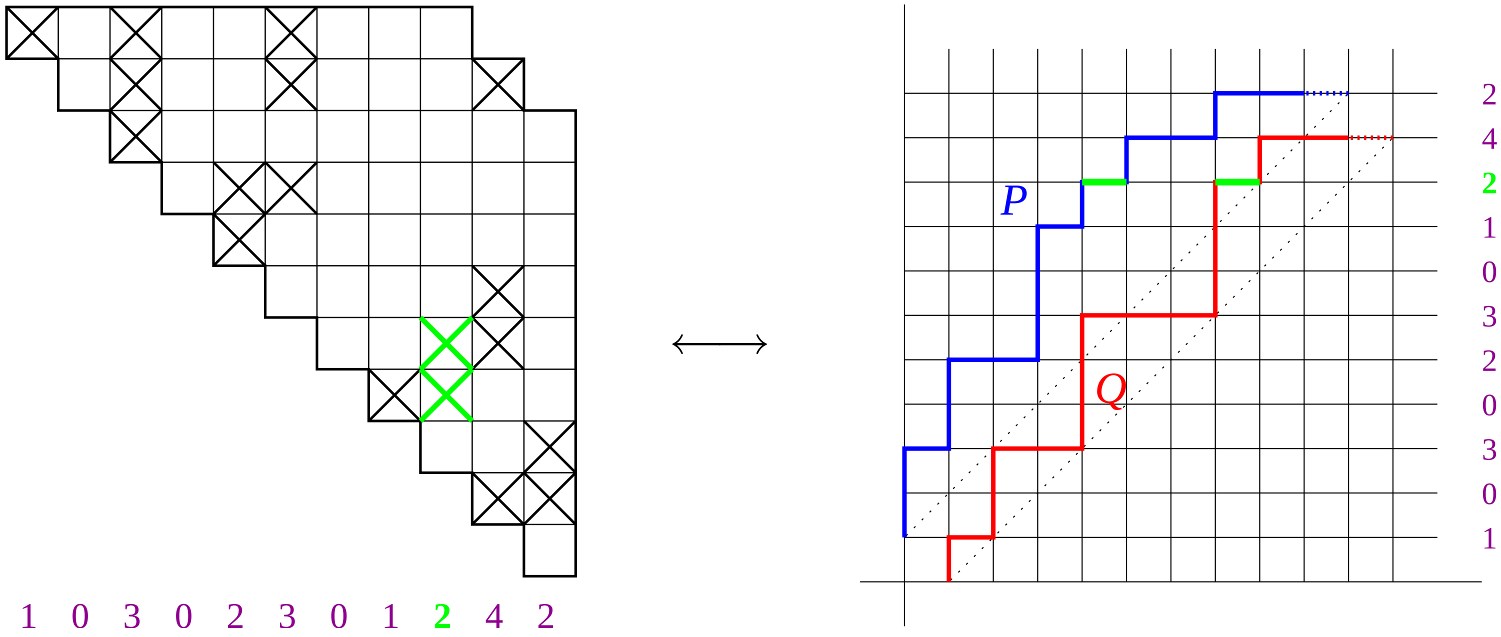


$\longleftrightarrow$



- The number of crosses in each column of the 2-triangulation equals the number of east steps at each level of the pair of Dyck paths (not counting the last east step of P and Q).

Some properties of the bijection



- The number of crosses in each column of the 2-triangulation equals the number of east steps at each level of the pair of Dyck paths (not counting the last east step of P and Q).
- Splitting a column in the 2-triangulation is equivalent to splitting a level of the pair of paths into two.

Generalization to arbitrary k?

Open problem:

Is there an analogous bijection between k-triangulations and k-tuples of non-crossing Dyck paths, for $k \geq 3$?

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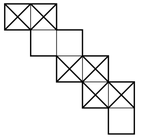
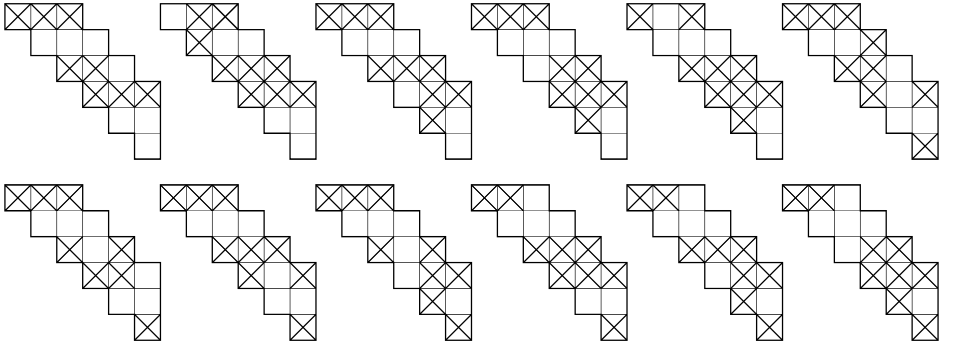


Partial progress:

- The same idea of splitting columns can be used to construct a generating tree for k-triangulations.



- However, it is not clear what is the corresponding operation to generate children of a k-tuple of Dyck paths that would give an isomorphic generating tree.