Maximal Brill–Noether loci via the gonality stratification

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Classical Brill–Noether theory

Let C be a smooth algebraic curve.

Definition

A g^r_d on C is a pair (A, V) of

- a line bundle $A \in \mathrm{Pic}^d(C)$ with $h^0(C, A) \geq r + 1$, and
- a subspace $V \subseteq H^0(C, A)$ of dimension $r + 1$.

A (basepoint free) g^r_d gives a map $C \to \mathbb{P}^r$ of degree d.

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When does C have a g^r_d?

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Brill–Noether loci

Theorem (Brill–Noether theorem)

A general curve $C \in \mathcal{M}_g$ *admits a* g^r_d *if and only if*

$$\rho(g,r,d) := g - (r+1)(g-d+r) \geq 0$$

Thus when $\rho(g,r,d) < 0$, the *Brill–Noether locus*

$$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \text{ admitting a } g^r_d\} \text{ is a subvariety of } \mathcal{M}_g.$$

Recall, the *gonality* of a curve is $\mathrm{gon}(C) := \min\{k \mid C \text{ admits a } g^1_k\}$.

Gonality stratification

A general curve in $\mathcal{M}^1_{g,k}$ has gonality k, and we have a stratification

$$\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \cdots \subset \mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor} \subset \mathcal{M}^1_{g,\lfloor\frac{g+3}{2}\rfloor} = \mathcal{M}_g$$

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Properties of Brill–Noether loci $\mathcal{M}^r_{g,d}$

- Can have multiple components of varying dimensions
- Each component has codimension at most $-\rho(g,r,d)$, the expected codimension
- $\operatorname{codim} \mathcal{M}^r_{g,d} = -\rho(g,r,d)$ for $-3 \leq \rho(g,r,d) \leq -1$
- Irreducible for $\rho = -1$ and distinct for $\rho = -1, -2$
- When $\rho(g,r,d)$ is not too negative ($\rho \geq -g+3$), have a component of expected codimension
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Refined Brill–Noether theory

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What linear systems does a "general" curve $C \in \mathcal{M}^r_{g,d}$ admit?

Theorem (Pflueger, Jensen–Ranganathan)

A general curve C of gonality k admits a g^r_d if and only if

$$0 \leq \rho_k(g,r,d) := \max_{0 \leq \ell \leq \min\{r, g-d+r-1\}} \rho(g, r-\ell, d) - \ell k.$$

Considering "general" $C \in \mathcal{M}^r_{g,d}$, refined Brill–Noether theory can be rephrased in terms of (non-)containments of Brill–Noether loci.

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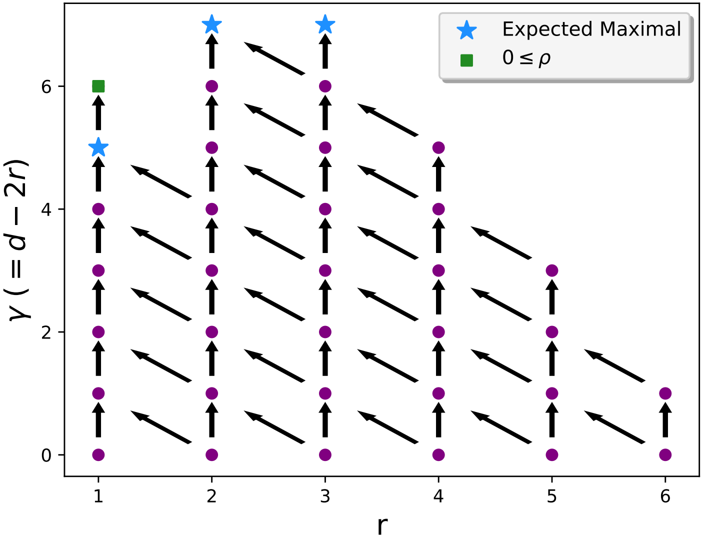
We have trivial containments

- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ by adding a basepoint
- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ by subtracting a non-basepoint

Brill–Noether loci in genus 14



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Definition

$\mathcal{M}^r_{g,d}$ is *expected maximal* if $d \leq g-1$ (up to Serre duality) and

- $\rho(g,r,d) < 0$,
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For each $1 \leq r \leq \lfloor \sqrt{g} - \frac{1}{2} \rfloor$, there is one expected maximal Brill–Noether locus with $d = d_{max}(g,r) := r + \lceil \frac{gr}{r+1} \rceil - 1$.
We write $\mathcal{M}^r_g := \mathcal{M}^r_{g,d_{max}(g,r)}$.

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Maximal Brill–Noether loci

Conjecture (Auel–H.)

For $g \geq 3$, except $g = 7, 8, 9$, the expected maximal Brill–Noether loci are maximal.

That is, for every pair of expected maximal loci there is some curve $C \in \mathcal{M}_g^r$ but $C \notin \mathcal{M}_g^s$.

In genus $7, 8, 9$, there are non-trivial containments:
$\mathcal{M}^2_{7,6} \subseteq \mathcal{M}^1_{7,4}$, $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$. [Larson, Mukai]

The conjecture holds in many cases:

- $g \leq 20, 22, 23$ [Farkas, Lelli-Chiesa, Auel–H., Auel–H.–Larson]
- $g+1$ or $g+2 \in \{\mathrm{lcm}(1, \ldots, n) \mid n \geq 4\}$ (all expected maximal BN loci have same $\rho \in \{-1, -2\}$) [Eisenbud–Harris, Choi–Kim–Kim]

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Distinguishing BN loci via gonality stratification

Definition

$$\kappa(g,r,d) := \max\{k \mid \mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}\}$$

$2 \leq \kappa(g,r,d)$: hyperelliptic curves have all g^r_ds (via trivial containments).

$\kappa(g,r,d) \leq \lfloor \frac{g+3}{2} \rfloor$: $\mathcal{M}^1_{g,\lfloor \frac{g+3}{2} \rfloor} = \mathcal{M}_g$.

$\kappa(8,2,7) = 4$

$\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$ (genus 8 counterexample) and $\mathcal{M}^1_{8,5} = \mathcal{M}_8$ so $\mathcal{M}^1_{8,5} \not\subseteq \mathcal{M}^2_{8,7}$.

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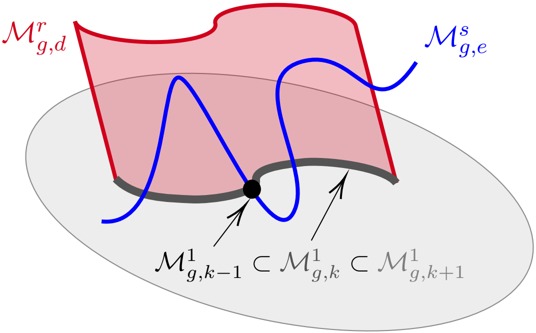
Proposition

Suppose $\kappa(g,r,d) > \kappa(g,s,e)$, *then* $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.

$$\kappa(g,r,d) = k > \kappa(g,s,e) = k-1$$



A general curve of gonality k is contained in $\mathcal{M}^r_{g,d}$, but not in $\mathcal{M}^s_{g,e}$.

$\kappa(g, r, d)$

By the refined Brill–Noether theory for curves of fixed gonality,

$$\kappa(g, r, d) = \max\{k \mid \rho_k(g, r, d) \geq 0\}.$$

$$\rho_k(g, r, d) = \max_{0 \leq \ell \leq \min\{r, g-d+r-1\}} \rho(g, r, d) + (g - k - d + 2r + 1)\ell - \ell^2,$$

which ranges over upside down parabolas.

Theorem (Auel–H.–Larson)

Let $d \leq g - 1$, then

$$\kappa(g, r, d) = \begin{cases} \lfloor d/r \rfloor & \text{if } g + 1 > d + \lfloor d/r \rfloor \\ g + 1 - d + 2r + \lfloor -2\sqrt{-\rho(g, r, d)} \rfloor & \text{else.} \end{cases}$$

Moreover, for expected maximal loci with $r \geq 2$, we always have $\kappa\left(\mathcal{M}^r_g\right) = g + 1 - d_{max}(g, r) + 2r + \lfloor -2\sqrt{-\rho} \rfloor$.

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Moreover, for expected maximal loci with $r \geq 2$, *we always have* $\kappa\left(\mathcal{M}_g^r\right) = g + 1 - d_{max}(g, r) + 2r + \lfloor -2\sqrt{-\rho} \rfloor$.

Simple proofs of non-containments of Brill–Noether loci

Theorem (Auel–H.)

For $g \neq 8$, $\mathcal{M}^1_g$ is maximal.

Compute $\kappa(\mathcal{M}^1_g) > \kappa(\mathcal{M}^r_g)$, hence $\mathcal{M}^1_g \not\subseteq \mathcal{M}^r_g$.

We obtain a new proof that Brill–Noether loci with $\rho = -1$ are distinct.

Theorem (Auel–H.–Larson)

For two expected maximal BN loci, if $\rho(g, r, d) = \rho(g, s, e)$ then we have $\mathcal{M}^r_g \not\subseteq \mathcal{M}^s_g$ or the other non-containment.

ρ and $d - 2r$ identify Brill–Noether loci up to Serre duality. Now use $\kappa\left(\mathcal{M}^r_g\right) = g + 1 - d_{max}(g, r) + 2r + \lfloor -2\sqrt{-\rho} \rfloor$.

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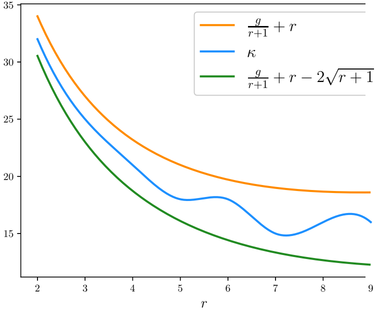
We obtain a new proof that Brill–Noether loci with $\rho = -1$ are distinct.

Theorem (Auel–H.–Larson)

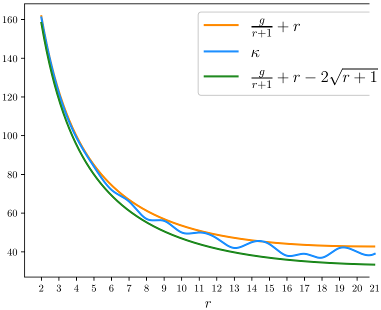
For two expected maximal BN loci, if $\rho(g,r,d) = \rho(g,s,e)$ then we have $\mathcal{M}^r_g \not\subseteq \mathcal{M}^s_g$ or the other non-containment.

ρ and $d - 2r$ identify Brill–Noether loci up to Serre duality. Now use $\kappa\left(\mathcal{M}^r_g\right) = g + 1 - d_{max}(g,r) + 2r + \lfloor -2\sqrt{-\rho} \rfloor$.

Bounds on $\kappa(g, r, d)$



$g = 96$



$g = 479$

Non-containments of Brill–Noether loci

Theorem (Auel–H.–Larson)

Fix $r \geq 2$. If $g \geq 4(r+1)^{5/2} + (r+1)^2 + 2(r+1)^{3/2}$, then $\kappa(\mathcal{M}_g^r) > \kappa(\mathcal{M}_g^s)$ for all $s > r$. In particular, $\mathcal{M}_g^r \not\subseteq \mathcal{M}_g^s$.

For each r, there exists a smallest $G(r)$ such that $\kappa(\mathcal{M}_g^r) > \kappa(\mathcal{M}_g^s)$:

r	2	3	4	5	6	7	8	9	10
$G(r)$	28	50	96	140	232	306	390	561	684

Fixing r, to prove that $\mathcal{M}_g^r$ is always maximal, it remains to check $\mathcal{M}_g^r \not\subseteq \mathcal{M}_g^q$ for $q < r$, and all non-containments for $g < G(r)$.

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Non-containments of Brill–Noether loci

Corollary (Auel–H.–Larson)

Except for $g = 7, 9$, and possibly $g = 24, 27$, the expected maximal Brill–Noether locus $\mathcal{M}^2_g$ is maximal.

To show $\mathcal{M}^2_g \not\subseteq \mathcal{M}^1_g$, we use K3 surfaces to exhibit a curve with a $g^2_{d_{max}(g,2)}$ and generic gonality.

Proposition (Auel–H.–Larson)

For any $g \geq 14$, $\mathcal{M}^r_g \not\subseteq \mathcal{M}^1_g$ for all expected maximal Brill–Noether loci with $r \geq 2$.

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Thank You!

Questions?