Maximal Brill-Noether loci

Parts joint with Asher Auel & Hannah Larson.

↳ classical Brill-Noether theory.

Study linear systems on curves.

C sm. curve.

Defn A g^r_d on C is a pair

$L \in \mathrm{Pic}^d(C)$ w/ $h^0(L) \geq r+1$, and

$V \subseteq H^0(L)$ of rank r.

$\rightsquigarrow$ gives a map $C \longrightarrow \mathbb{P}^r$ of degree d.

Q: when does C have a g^r_d?

Brill-Noether theorem A general curve C of genus g admits a g^r_d iff

$$\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0.$$

E.g./ Not every curve of genus 3 is hyperelliptic.

$$\rho(3,1,2) = 3-(2)(2) = -1.$$

The <u>gonality</u> of a curve is

$\mathrm{gon}(C) = \min\{k \mid C \text{ admits a } g^1_k\}$.

By the BN theorem,

$\mathrm{gon}(C) \leq \lfloor \frac{g+3}{2} \rfloor$, moreover,

for general C, $\mathrm{gon}(C) = \lfloor \frac{g+3}{2} \rfloor$.

<u>Defn</u> The <u>Brill-Noether loci</u> are

$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \text{ admitting a } g^r_d\}$

when $\rho(g,r,d) < 0$, $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}_g$ is a proper subvariety.

- $\mathcal{M}^r_{g,d}$ can have multiple components, of different dimensions.

- Each component has codimension at most $-\rho$, the expected codim.

- codim $\mathcal{M}^r_{g,d} = -\rho$ for $-3 \leq \rho \leq -1$

- $\mathcal{M}^r_{g,d}$ irred. when $\rho = -1$

  ↳ BN divisors used in study of Kodaira dimension of $\mathcal{M}_g$

↳ Refined BN Theory

Thm [Pflueger, Jensen-Ranganathan]

- C general of gonality k, then C has a g^r_d iff

$$\rho_k(g,r,d) = \max_{0 \leq \ell \leq r'} \rho(g, r-\ell, d) - \ell k \geq 0.$$

$$(r' = \min\{r, g-d+r-1\})$$

Q: When does (a "general") $C \in \mathcal{M}^r_{g,d}$ admit a g^s_e?

- How do BN loci stratify $\mathcal{M}_g$?

Trivial containments:

- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$

Q what are the maximal BN loci?

Defn $\mathcal{M}^r_{g,d}$ is expected max'l if

$$\rho(g,r,d) < 0, \qquad \left(d = r + \left\lceil \frac{gr}{r+1} \right\rceil - 1\right).$$
$$\rho(g,r,d+1) \geq 0, \text{ and}$$
$$\rho(g,r-1,d-1) \geq 0.$$

Conj [Auel-H.] For any $g \geq 3$, except 7, 8, 9, the expected max'l BN loci are max'l.

known:
- for ∞-ly many g
- for $g \leq 23$
- many non-containments known

what happens in genus 7, 8, 9?
- secant varieties give non-trivial containments.

E.g./ genus 8

$\mathcal{M}^1_{8,4}$ $\mathcal{M}^2_{8,7}$ are the exp. maxe'l loci

Let A be a g^1_4, then $\omega_C - A = g^4_{10}$ gives $C \subseteq \mathbb{P}^4$, which will have a 3-secant line, giving a g^2_7. so $\mathcal{M}^1_{8,4} \subseteq \mathcal{M}^2_{8,7}$.

↳ Via gonality stratification. J.w. Asher Auel & Hannah Larson.

Defn

$$\kappa(g,r,d) = \max\{k \mid \mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}\}$$

[P, J-R]

$$= \max\{k \mid \rho_k(g,r,d) \geq 0\}.$$

E.g./ $\kappa(8,2,7) = 4$.

Prop If $\kappa(g,r,d) > \kappa(g,s,e)$, then

$$\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}.$$

Pf/ Since $\kappa = \kappa(g,r,d) > \kappa(g,s,e)$

so $\mathcal{M}^1_{g,\kappa} \not\subseteq \mathcal{M}^s_{g,e}$.

$$\begin{array}{ccc} \mathcal{M}^r_{g,d} & \not\subseteq & \mathcal{M}^s_{g,e} \\ \cup\!| & \cancel{\subseteq} & \\ \mathcal{M}^1_{g,\kappa} & & \end{array}$$

$\boxed{\Xi}$.

Prop If $d \le g-1$, $\mathcal{K}(g,r,d) = \begin{cases} \lfloor \frac{d}{r} \rfloor & ; \quad g+1 > \lfloor \frac{d}{r} \rfloor + d \\ g+1-d+2r+\lfloor -2\sqrt{-\rho} \rfloor & ; \quad \text{else} \end{cases}$

Focus on exp. max'l BN loci:

Thm For $g \geq 9$, $\mathcal{M}^1_{g, \lfloor \frac{g+1}{2} \rfloor} \not\subset \mathcal{M}^r_{g,d}$ $\quad \forall r \geq 2$ exp. max'l.

Pf/ $\mathcal{K}(g,r,d) < \lfloor \frac{g+1}{2} \rfloor$. $\boxtimes$.

Lemma: if $\rho(g,r,d) = \rho(g,s,e)$, then $\mathcal{K}(g,r,d) \neq \mathcal{K}(g,s,e)$.

Prop If $r, s \geq 2$, $\rho(g,r,d) = \rho(g,s,e)$ and $\mathcal{M}^r_{g,d}$, $\mathcal{M}^s_{g,e}$ are exp. max'l, then one non-cont. holds

Thm If $\rho(g,r,d) = \rho(g,s,e) = -1$ then $\mathcal{M}^r_{g,d} \not\subset \mathcal{M}^s_{g,e}$.
(and $\mathcal{M}^s_{g,e} \not\subset \mathcal{M}^r_{g,d}$.)

(**Fact**: $\mathcal{M}_{g,d}$ is irred. if $\rho = -1$ [Eisenbud, Harris])

Thm If $g-1$ or $g-2 \in \{\mathrm{lcm}(1,\dots,n) \mid n \geq 4\}$ then the max BN loci conj. holds.

Pf/ all max BN loci have same $\rho \in \{-1,-2\}$
if $\rho=-2$, known to be distinct [Choi, Kim, Kim] [3].

Lemma For $\mathcal{M}^s_{g,e}$ exp. max'l,

$$\frac{g}{s+1} + s - 2\sqrt{s+1} < K(g,s,e) \leq \frac{g}{s+1} + s.$$

Thm $\exists\, G(r) \leq 4(r+1)^{5/2} + (r+1)^2 + 2(r+1)^{3/2}$ s.t. $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$ $\forall s > r$, $g \geq G(r)$ exp. max'l.

Thm For $g \geq 28$, $\mathcal{M}^2_{g,d}$ exp. max'l is max'l.

Pf/ $\mathcal{M}^2_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$ $\forall s \geq 3$ by Thm.

RTS $\mathcal{M}^2_{g,d} \not\subseteq \mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}$.

↳ Via K3 surfaces. j.w. Asher Auel.

Strategy: To show $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$,
find C w/ a g^r_d but no g^s_e
on a K3.

① C with a g^r_d:

Let (S,H) be a polarized K3 surface
with

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}.$$

Prop $C \in |H|$ sm. irred has gonality
$\lfloor \frac{g+3}{2} \rfloor$ and has a g^r_d
(might not be $L|_C$, but in nice
cases, it is.)

Cor $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ for $r \geq 2$ exp max'l.

② What if C has a g^s_e?

Idea: • Then $\exists\, M \in \mathrm{Pic}(S)$ w/ certain numerical properties (*)

• Show such M cannot exist.

(*)

<u>Donagi-Morrison Conj</u>:

If C has a g^s_e w/ $\rho < 0$, then $\exists\, M \in \mathrm{Pic}(S)$ s.t. $g^s_e \subseteq |M|\big|_C$ and M satisfies some numerical properties.

False in general [Lelli-Chiesa-Knutsen]

Bounded versions for $e \leq \underline{B}(\mathrm{gon}(C), g, \mathrm{Pic}(S))$

Known: $s=1$ [DM]

$s=2$ [Lelli-Chiesa]

$s=3$ [H]

<u>Proof idea</u>: Study Lazarsfeld-Mukai bundle $\underline{E}$ associated to g^s_e (it is unstable)

<u>Prop</u> If $N \subseteq E$ saturated line bundle w/ $h^0(N) \geq 2$, then $M = \det(E/N)$ works.

To find N:

"Morally"
Consider a destabilizing filtration $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell \subset E$ of E s.t. E_{i+1}/E_i stable, torsion-free, and $\mu(E_i/E_{i-1}) \geq \mu(E_{i+1}/E_i)$.

Show that if $\ell > 1$ & $\mathrm{rk}\, E_1 > 1$, then $c_2(E) >> 0$, and does not exist on S.

So $\exists\ N \subseteq E$, as desired.

slogan: $\mathrm{Pic}(S)$ controls which unstable LM bundles exist.