Brill–Noether special cubic fourfolds of discriminant 26

Richard Haburcak
(Joint with Asher Auel)

Dartmouth College

AMS Spring Central Sectional
April 16^{th}, 2023

Special cubic fourfolds

Let $\mathcal{C}$ be the 20-dimensional moduli space of smooth cubic fourfolds $X \subset \mathbb{P}^5$.

Let X be a cubic fourfold, we denote by

$$A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

the lattice of algebraic cycles.

Definition

X is *special* if $\operatorname{rk} A(X) \geq 2$.

That is, X contains an algebraic surface T not homologous to a complete intersection.

Hassett divisors

The special cubic fourfolds are a union of divisors Hassett $\mathcal{C}_d$ of cubic fourfolds such that $A(X)$ admits a primitive embedding preserving h^2 of a lattice of discriminant d.

$\mathcal{C}_d$ is nonempty if $d > 6$ and $d \equiv 0$ or $2 \pmod 6$. (e.g., 8, 12, 14, 18, 20, 24, 26, ...)

Labelled cubic fourfolds

More generally, one can consider *lattice polarized* cubic fourfolds.

Definition

Let Λ be a positive definite lattice with a distinguished element $h^2 \in \Lambda$ of norm 3.

A Λ-*polarized cubic fourfold*, (X, Λ), is a cubic fourfold X together with a fixed primitive isometric embedding $\Lambda \hookrightarrow A(X)$ preserving h^2.

Let $\mathcal{C}_\Lambda$ denote the moduli space of Λ-polarized cubic fourfolds.

By work of Laza and Looijenga, $\mathcal{C}_\Lambda$ is non-empty if and only if Λ admits a primitive embedding into $H^4(X, \mathbb{Z})$ and Λ has no *short roots* or *long roots*.

Cubic fourfold lattices

Let Λ be a positive definite lattice with distinguished element h^2 of norm 3.

Definition

We say $v \in \langle h^2 \rangle^\perp \subseteq \Lambda$ is a

- *short root* if $v^2 = 2$
- *long root* if $v^2 = 6$ and $v \pm h^2 \in 3\Lambda$.

A positive definite lattice Λ such that $\mathcal{C}_\Lambda$ is non-empty is called a *cubic fourfold lattice*.

Following Hassett's argument for $\mathcal{C}_d$ shows that $\mathcal{C}_\Lambda$ is a $(21 - \operatorname{rk} \Lambda)$-dimensional quasiprojective variety.

Associated polarized K3 surfaces

Let $d > 6$ divide $2(n^2+n+1)$, (e.g. 14, 26, 38, 42, $\dots$)

For a cubic fourfold X with a marking of discriminant d, there is an *associated* polarized K3 surface (S, H) of genus $g = \frac{d}{2} + 1$ such that there is a Hodge isometry

$$H^4(S, \mathbb{Z}) \supset K_d^\perp \cong H^\perp(-1) \subset \mathrm{Pic}(S)(-1).$$

This gives rise to an open immersion $\mathcal{C}_\Lambda \hookrightarrow \mathcal{K}_g$ of moduli spaces.

We extend this open immersion to Λ-polarized cubic fourfolds.

Lattice theory

Let

$$K_d = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & x \\ T & x & y \end{array} \subset \Lambda = \begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & x & a \\ T & x & y & b \\ J & a & b & c \end{array}$$

with K_d be a positive definite lattice of discriminant d and Λ a cubic fourfold lattice.

Lemma

Suppose $\gcd(3, h^2.T) | h^2.J$ *and* $-3 \in (\mathbb{Z}/d\mathbb{Z})^{\times 2}$*.*
Then, up to isometry, there is a unique rank 2 *even indefinite lattice*

$$\sigma(\Lambda) = \begin{array}{c|cc} & H & M \\ \hline H & d & \alpha \\ M & \alpha & \beta \end{array}$$

with discriminant $-\operatorname{disc}(\Lambda)$ *such that* $K_d^{\perp} \cong H^{\perp}(-1)$*.*

Standard form of lattices

Corollary

Since $\gcd(3, h^2.T) | h^2.J$*, we can choose* $a = 0$*, i.e., there is a basis of* Λ *with respect to which* Λ *has Gram matrix*

$$\begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & x & 0 \\ T & x & y & b \\ J & 0 & b & c \end{array}$$

for some integers $0 \le b \le d/2$ *and* $c > \max(2, 3b^2/d)$ *even.*

$\mathrm{Pic}(S)$ of associated K3

The standard form of Λ gives us an algorithm for computing $\sigma(\Lambda)$.

For fixed K_d,

$$\Lambda = \begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & x & 0 \\ T & x & y & b \\ J & 0 & b & c \end{array} \text{ is determined by } (b, c).$$

$$\sigma(\Lambda) = \begin{array}{c|cc} & H & M \\ \hline H & d & \alpha \\ M & \alpha & \beta \end{array} \text{ is determined by } (\alpha, \beta).$$

We can recover Λ with the fixed embedding of K_{26} from $\sigma(\Lambda)$.

Moduli Summary

Theorem (Auel–H.)

Let $d = 2g - 2$ have a unique odd prime divisor.
Let $\Lambda = \langle h^2, T, J \rangle$ be a rank 3 cubic fourfold lattice with a fixed primitive embedding of K_d preserving h^2 such that $\gcd(3, h^2.T) | h^2.J$. Then there exists an open immersion $\mathcal{C}_\Lambda \hookrightarrow \mathcal{K}_{\sigma(\Lambda)}$ of moduli spaces and a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{K_d} & \hookrightarrow & \mathcal{K}_g \\ \uparrow & & \uparrow \\ \mathcal{C}_\Lambda & \hookrightarrow & \mathcal{K}_{\sigma(\Lambda)} \end{array}$$

where the vertical arrows are the forgetful maps and the top arrow is the open immersion constructed by Hassett.

Cubic fourfolds of discriminant 26

$$K_{26} = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 7 \\ T & 7 & 25 \end{array}$$

A cubic fourfold containing a 3-nodal septic scroll is K_{26}-polarized, as does the general $X \in \mathcal{C}_{26}$.

We focus on cubic fourfolds that are Λ-polarized, where Λ has a fixed primitive embedding $K_{26} \hookrightarrow \Lambda$.

$$\{\text{What does } \Lambda \text{ look like?}\} \longleftrightarrow \{\text{What does } \mathrm{Pic}(S) \text{ look like?}\}$$

Brill–Noether special cubic fourfolds

Definition (Mukai)

A polarized K3 surface (S, H) of genus g, $(H^2 = 2g - 2)$, is *Brill–Noether special* if there is a nontrivial $J \neq H \in \mathrm{Pic}(S)$ such that

$$g - h^0(S, J)h^0(S, H - J) < 0.$$

Else (S, H) is called *Brill–Noether general*.

Definition (Auel)

A special cubic fourfold X is *Brill–Noether special* if it has an associated polarized K3 surface (S, H) that is Brill–Noether special.

Brill–Noether special K3s of genus 14

Definition

A curve C of genus g is *Brill–Noether special* if it has a line bundle A such that

$$\rho(g,r,d) = \underbrace{g}_{\text{genus}(C)} - \underbrace{(r+1)}_{h^0(C,A)}\underbrace{(g-d+r)}_{h^0(C,\omega_C - A)} < 0.$$

In genus 8, a curve is BN special if and only if it has a g^2_7. In higher genus, this becomes more complicated.

Theorem (Auel–H.)

Let (S,H) be a polarized K3 surface of genus 14. Then (S,H) is Brill–Noether special if and only if a smooth irreducible curve $C \in |H|$ is Brill–Noether special.

Genus 8

Theorem (Mukai, Lelli-Chiesa, Auel)

A cubic fourfold of discriminant 14 is Brill–Noether special if and only if the associated K3 satisfies that $C \in |H|$ has a g^2_7. Moreover, such cubic fourfolds contain 2 disjoint planes.

Proposition (Auel)

A cubic fourfold is pfaffian if and only if it has a discriminant 14 marking whose associated K3 is Brill–Noether general.

Theorem (Auel)

The complement of the pfaffian locus is contained in the locus of cubic fourfolds containing 2 disjoint planes.

Lattices of Brill–Noether special K3s of genus 14

Let $\mathcal{K}_g$ be the moduli space of primitively quasi-polarized K3 surfaces of genus g.
The Noether–Lefschetz divisor $\mathcal{K}^r_{g,d} \subset \mathcal{K}_g$ parameterizes K3 surfaces with a specific lattice polarization

$$\Lambda^r_{g,d} := \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array} \subseteq \mathrm{Pic}(S).$$

Theorem (Greer–Li–Tian)

(S,H) is Brill–Noether special if and only if $(S,H) \in \mathcal{K}^r_{g,d}$ where

- $2 \leq d \leq g-1$,
- $\Delta(g,r,d) := \mathrm{disc}(\Lambda^r_{g,d}) = 4(r-1)(g-1) - d^2 < 0$,
- *and* $\rho(g,r,d) < 0$.

Associated cubic fourfold lattices

For each of these lattices, we recover a lattice Λ such that $\sigma(\Lambda) = \Lambda^r_{g,d}$, excluding those with roots.

Theorem (Auel–H.)

Let X be a Brill–Noether special cubic fourfold of discriminant 26, and (S, H) the associated polarized K3 of genus 14. Then $\mathrm{Pic}(S)$ has a primitive embedding of one of the following lattices:

- $\Lambda^1_{14,6}$ $(\gamma(C) = 4)$
- $\Lambda^2_{14,9}$ $(\gamma(C) = 5)$
- $\Lambda^2_{14,10}$ $(\gamma(C) = 6)$
- $\Lambda^2_{14,11}$, $\Lambda^3_{14,13}$ $(\gamma(C) = 6)$

In particular, for the general such K3 surface, $\gamma(C) = 4, 5,$ or 6, and C has no other Brill–Noether special line bundles of Clifford index $\leq \gamma(C)$.

Interesting Overlaps

Given $\Lambda^r_{g,d} = \sigma(\Lambda)$ on the list above, we look for which other markings Λ admits.

$\Lambda^1_{14,6}$

$$\begin{array}{c|cc} & H & L \\ \hline H & 26 & 6 \\ L & 6 & 0 \end{array} \text{ gives the lattice } \begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & 7 & 0 \\ T & 7 & 25 & 12 \\ J & 0 & 12 & 18 \end{array}$$

- $\langle h^2, -2h^2 + T - J \rangle$ gives a discriminant 8 marking.
- $\langle h^2, 6h^2 - 2T + J \rangle$ gives a discriminant 14 marking.

We note that such cubic fourfolds form a component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, due to Auel–Bernardara–Bolognesi–Várilly-Alvarado, and the general one has non-trivial Clifford invariant $\beta \in \mathrm{Br}(S)$.

Interesting Overlaps

$\Lambda^2_{14,9}$

$$\begin{array}{c|cc} & H & L \\ \hline H & 26 & 9 \\ L & 9 & 2 \end{array} \text{ gives the lattice } \begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & 7 & 0 \\ T & 7 & 25 & 5 \\ J & 0 & 5 & 4 \end{array}$$

- $\langle h^2, -2h^2 + T - J \rangle$ gives a discriminant 8 marking.
- $\langle h^2, -h^2 + T - 2J \rangle$ gives a discriminant 14 marking.

We note that such cubic fourfolds form a component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, due to Auel–Bernardara–Bolognesi–Várilly-Alvarado, and have trivial Clifford invariant, hence are rational.

Interesting Overlaps

Theorem (Auel–H.)

Let X be a Brill–Noether special cubic fourfold of discriminant 26, then $X \in \mathcal{C}_8$.

We note that general Brill–Noether special cubic fourfolds of discriminant 26 admit no $\mathcal{C}_{42}$ marking.

Summary

$$\{\text{Rank } 3 \text{ cubic fourfold lattices}\} \longleftrightarrow \left\{\substack{\text{Rank } 2 \text{ K3 lattices} \\ \text{Brill–Noether special K3 surfaces}}\right\}$$

$$\Updownarrow$$

$$\{\text{Brill–Noether special curves}\}$$

Summary

{Rank 3 cubic fourfold lattices} ⟷ {Rank 2 K3 lattices / Brill–Noether special K3 surfaces}

{Rank 3 cubic fourfold lattices} ∼∼∼ ??? ∼∼∼ {*Lazarsfeld–Mukai Bundles*}

{Brill–Noether special K3 surfaces} ⇕ {*Lazarsfeld–Mukai Bundles*} ⇕ {Brill–Noether special curves}

Thank You!

Questions?