Recovering Complex Fourier Transformation Phases

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August 17, 2017

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Parallel Problem

- Suppose we are given a and b
- You know a and b are reals
- You know that $a + b = 4$
- You know nothing else about and b
- Find a and b

- Suppose $a + b = 4$
- What if a and b are rationals instead?
- Integers?
- Can you find a or b?

- Q: Why is it impossible recover a or b?
- A: Because there are multiple ways to choose two reals, rationals, or integers which satisfy a+b=4

- There is ambiguity in the data.
- How can we overcome such ambiguity?
- We can't... unless

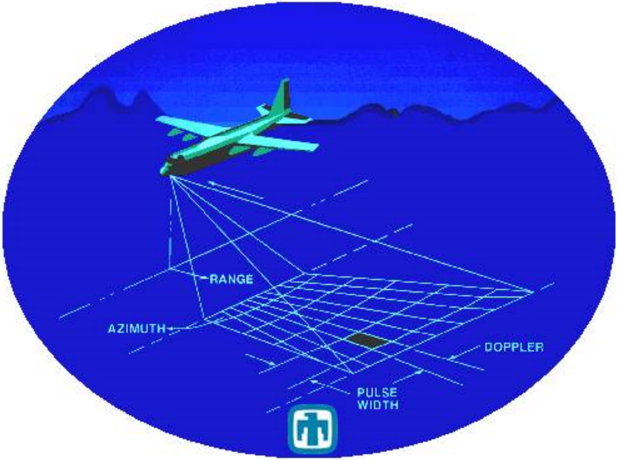
- Unless we make assumptions about the data
- "Weak" assumptions like a and b are real, rational, or integer do not help

- We need to make strong assumptions.
- e.g assume a=2. Then, b=2, and we know a and b

The Actual Problem

- SAR $\rightarrow$ complex Fourier transform data (FTD)
- Magnitude is sparse, like in real-valued Fourier transform data
- Problem: phase is not necessarily sparse
- No expectations about the phase of the FTD

- Bigger problem: see blurring errors on the phase of the FTD
- One such error in SAR: the collected FTD is of the form
  - $F(w_j, \theta_k) = e^{-iw_j\phi(\theta_k)} f(w_j, \theta_k)$
- w represents frequency
- θ represents the azimuthal angle
- F is the blurred Fourier transform
- f is the true Fourier transform data
- $\phi(\theta_k)$ is unknown
- j=1, 2, ..., r, and k=1,2, ..., p



Source: sandia.gov

Sovling the Actual Problem

- $|F(w_j, \theta_k)| = |f(w_j, \theta_k)|$
- Only need to worry about recovering $phase(f(w_j, \theta_k))$, for now
- System is now of the form:
  - $phase(F(w_j, \theta_k)) = -w_j\phi(\theta_k) + phase(f(w_j, \theta_k)) + 2\pi N$
- N is a normalizing constant
- Take N to be zero for now

- Let $A = \begin{bmatrix} phase(F(w_1,\theta_1)) & phase(F(w_1,\theta_2)) & \dots & phase(F(w_1,\theta_p)) \\ phase(F(w_2,\theta_1)) & phase(F(w_2,\theta_2)) & \dots & phase(F(w_2,\theta_p)) \\ \vdots & \vdots & \ddots & \vdots \\ phase(F(w_r,\theta_1)) & phase(F(w_r,\theta_2)) & \dots & phase(F(w_r,\theta_p)) \end{bmatrix}$
- Let $B = \begin{bmatrix} phase(f(w_1,\theta_1)) & phase(f(w_1,\theta_2)) & \dots & phase(f(w_1,\theta_p)) \\ phase(f(w_2,\theta_1)) & phase(f(w_2,\theta_2)) & \dots & phase(f(w_2,\theta_p)) \\ \vdots & \vdots & \ddots & \vdots \\ phase(f(w_r,\theta_1)) & phase(f(w_r,\theta_2)) & \dots & phase(f(w_r,\theta_p)) \end{bmatrix}$

- Let $W = \begin{bmatrix} -w_1 & -w_1 & \dots & -w_1 \\ -w_2 & -w_2 & \dots & -w_2 \\ \vdots & \vdots & \ddots & \vdots \\ -w_r & -w_r & \dots & -w_r \end{bmatrix}$
- Let $P = \begin{bmatrix} \phi(\theta_1) & 0 & 0 & \dots & 0 \\ 0 & \phi(\theta_2) & 0 & \dots & 0 \\ 0 & 0 & \phi(\theta_3) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \phi(\theta_p) \end{bmatrix}$.

- System is now A=WP+B
- Set of possible values of the FTD B which can reproduce the collected data A is
  - $X^* = \{A - WP\}$

- W and A are completely determined
- P is completely unknown
- The set of possible P forms a subspace of $\mathbf{R}^{pxp}$ of dimension p, as does the set of possible WP
- So there are uncountably many solutions for B
- Then, $X^* = \{A - WP\}$ is an affine space of dimension, or size, p
- We will use such a measure of size as the quality of an estimate of B
- Removing the assumption that the normalizer N is nonzero will never decrease the size of the solution space

- Each element of X^* could be have generated the collected data
- There is no way to find the true solution for the FTD, B

- Q: What to do?
- A: Make Strong Assumptions

No Initial Error

- Assume there is no error for θ_1
- This case is simple to analyze and is realistic
- Consider $X^* = \{A - WP\}$ arising in the assumption-less case
- Reminders:
    - $A_{jk} = phase(F(w_j, \theta_k))$
    - $W_{jk} = w_j$,
    - $P_{jj} = \phi(\theta_j)$
    - Take the normalizer N to be zero
    - Removing the assumption that the normalizer is zero will never decrease the size of the space of the possible solutions

- We impose exactly one constraint on X^*: that $P_{11} = 0$
- Then, X^* becomes an affine space of degree p-1
- Better than assuming nothing, but not by much
- We only decrease the size of the solution space by 1!

Azimuth Independent FTD

- 1^{st} strong assumption: assume FTD is independent of the azimuthal angle, θ
- The system is then:
  - $f_k(w) = e^{-iw\phi_k} f(w)$,
- $\phi_k = \phi(\theta_k)$
- $f_k(w) = F(w, \theta_k)$
- $f(w) = f(w, \theta_k)$

- Again, only the phases need be considered
- Then, we have:
  - $phase(f_k(w)) = -iw\phi_k + phase(f(w)) + 2\pi N$
- N is a normalizer
- Assume for now that N is 0

- Then, for any given ϕ_k, there corresponds a system of p equations:
    - $phase(f(w_1)) - w_1\phi_k = phase(f_k(w_1))$
    - $phase(f(w_2)) - w_2\phi_k = phase(f_k(w_2))$
    - ...
    - $phase(f(w_r)) - w_r\phi_k = phase(f_k(w_r))$

- System on the last slide can be expressed as Ax=b^k
  - $A = \begin{bmatrix} 1 & 0 & \dots & 0 & -w_1 \\ 0 & 1 & \dots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -w_r \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$
  - $x = (x_1, x_2, ..., x_r, \phi_k)^t$
  - $b^k = (b_1^k, b_2^k, ..., b_r^k, 0)^t$
  - $x_n = phase(f(w_n))$
  - $b_n^k = phase(f_k(w_n))$, for n=1,2, ..., r

- $Ker(A) = a(w_1, w_2, ..., w_r, 1)$
- $a \in \mathbf{R}$
- Particular solution is
  - $x = (w_1\phi_k + b_1^k, w_2\phi_k + b_2^k, ..., w_r\phi_k + b_r^k, \phi_k)^t$
- So, solution set is
  - $x_{\phi_k}^* = \{(b_1^k, b_2^k, ..., b_r^k, 0)^t + a(w_1, w_2, ..., w_r, 1) : a \in \mathbf{R}\}$
- The solution set for the FTD is then
  - $X^* = \{(b_1^k, b_2^k, ..., b_r^k)^t + a(w_1, w_2, ..., w_r) : a \in \mathbf{R}\}$

- X^* is then an affine space with dimension 1
- Compare to the the dimension of the solution space–p–if azimuthal independence is not assumed
- What if normalizer N is considered?
- The dimension p of the solution space will not decrease

- There are still infinitely many solution
- We need one solution
- The assumption of Azimuth-independent FTD is most likely unrealistic
- We need to do better!

No Initial Error and Azimuth Independent FTD

- Assume no initial error *and* Azimuth Independent FTD
- Then, $f(w) = F(w, \theta_1)$!
- The system is solved
- Again, the assumption of Azimuth Independent FTD is most likely unrealistic
- Still need to do better!

Frequency-independent FTD

- Assume that the FTD is effectively frequency-independent
- This is most likely more realistic than assuming the FTD is Azimuth-independent

- There are r equations involving $\phi(\theta_k)$ for each k, yielding the system of equations:
  - $phase(F(w_1, \theta_k)) = w_1\phi(\theta_k) + phase(f(\theta_k)) - 2pi\lfloor \frac{w_1\phi(\theta_k) + phase(f(\theta_k))}{2pi} \rfloor$
  - $phase(F(w_2, \theta_k)) = w_2\phi(\theta_k) + phase(f(\theta_k)) - 2pi\lfloor \frac{w_2\phi(\theta_k) + phase(f(\theta_k))}{2pi} \rfloor$

$$\ldots$$

  - $phase(F(w_r, \theta_k)) = w_r\phi(\theta_k) + phase(f(\theta_k)) - 2pi\lfloor \frac{w_r\phi(\theta_k) + phase(f(\theta_k))}{2pi} \rfloor,$

- Provided that the normalizing terms, $2pi\lfloor \frac{w_j\phi(\theta_k) + phase(f(\theta_k))}{2pi} \rfloor$, can be removed, the system can be solved
- Removing such normalizing terms is known as *phase unwrapping*.

- There is a famous algorithm described by Kazuyoshi Itoh in his paper *Analysis of Phase Unwrapping Algorithm*
- The algorithm starts with the smallest value of a sequence of unwrapped phases
- Then, if the next value of the sequence is larger than π, subtract 2π
- If the next value is smaller than $-\pi$, add 2π

- Itoh's algorithm has some drawbacks:
  - The differences must be between $-\pi$ and π
  - The algorithm poorly handles noise.
- Nevertheless, I was able to successfully implement Itoh's algorithm on noiseless frequency-independent FTD assuming that $-\pi < max_{j\in(1,r-1)}\{|w_j - w_{j+1}|\}\phi(\theta_k) < \pi$
- If a bound on $\phi(\theta_k)$ is known, then the minimum resolution $max_{j\in(1,r-1)}\{|w_j - w_{j+1}|\}$ which permits the use of Itoh's algorithm may be applied
- i.e we have a resolution problem
- There are other algorithms, however, which all have their drawbacks

Suggestions for Further Study

- Further investigate algorithms for solving the one-dimensional phase unwrapping problem
- Investigate which other reasonable assumptions admit recovery of FTD data

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