1 Evaluate $\int x\cos^2 3x\,dx$

Solution: First rewrite $\cos^2 3x$ using the half-angle formula:

$$\int x\cos^2 3x\,dx = \int x\left(\frac{1+\cos 6x}{2}\right)dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 6x\,dx.$$

Now use integration by parts to evaluate $\int x\cos 6x\,dx$, setting $u = x$ and $dv = \cos 6x\,dx$, which makes $du = dx$ and $v = \sin 6x/6$:

$$\begin{aligned} \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 6x\,dx &= \frac{x^2}{4} + \frac{x\sin 6x}{12} - \int \frac{\sin 6x}{12}\,dx \\ &= \frac{x^2}{4} + \frac{x\sin 6x}{12} + \frac{\cos 6x}{72}\,dx + C \end{aligned}$$

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2 Evaluate $\int e^{2x}\sin x\,dx$.

Solution: We use integration by parts twice. Set $I = \int e^{2x}\sin x\,dx$. Now, using integration by parts with $u = e^{2x}$ and $dv = \sin x\,dx$ (the other choice of u and dv works just as well), so $du = 2e^{2x}\,dx$ and $v = -\cos x$, we have

$$I = -e^{2x}\cos x - \int -2e^{2x}\cos x\,dx.$$

Using integration by parts again with $u = -2e^{2x}$ and $dv = \cos x\,dx$, we get

$$\begin{aligned} I &= -e^{2x}\cos x - \left(-2e^{2x}\sin x - \int -4e^{2x}\sin x\,dx\right) \\ &= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx + C \\ &= -e^{2x}\cos x + 2e^{2x}\sin x - 4I + C. \end{aligned}$$

Solving this equation for I, we see that

$$I = \frac{-e^{2x}\cos x + 2e^{2x}\sin x}{5} + C.$$

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3 Could you in principle compute $\int x^{10^{10}} e^x \, dx$, and if so, how?

Solution: Yes, using integration by parts 10^{10} times, each time setting u equal to the polynomial and letting $dv = e^x \, dx$.

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4 Evaluate $\int \sin^3 x \cos^4 x \, dx$.

Solution: Since the power of sine is odd, we convert 2 of the sines into cosines using $\sin^2 x + \cos^2 x = 1$, so $\sin^2 x = 1 - \cos^2 x$:

$$\int \sin^3 x \cos^4 x \, dx = \int \sin x \left(1 - \cos^2 x\right) \cos^4 x \, dx.$$

Now we make a u-substitution, setting $u = \cos x$ and $du = -\sin x \, dx$:

$$\begin{aligned} \int \sin x \left(1 - \cos^2 x\right) \cos^4 x \, dx &= -\int (1 - u^2) u^4 \, du \\ &= -\int u^4 - u^6 \, du \\ &= -\frac{u^5}{5} + \frac{u^7}{7} + C \\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C. \end{aligned}$$

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5 Evaluate $\int \sec^4 x \tan^4 x \, dx$.

Solution: Since the power of secant is even, we save a $\sec^2 x$ and convert the other secants to tangents using the identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \sec^4 x \tan^4 x \, dx = \int \sec^2 x \left(1 + \tan^2 x\right) \tan^4 x \, dx.$$

Now we make a u-substitution, setting $u = \tan x$ and $du = \sec^2 x \, dx$:

$$\begin{aligned} \int \sec^2 x \left(1 + \tan^2 x\right) \tan^4 x \, dx. &= \int (1 + u^2) u^4 \, du \\ &= \int u^4 + u^6 \, du \\ &= \frac{u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C. \end{aligned}$$

6 What substitution would you use to evaluate $\int x^3\sqrt{16+x^2}\,dx$?

Solution: We would like to simplify the radical $\sqrt{16+x^2}$ using the identity $16+16\tan^2\theta = 16\sec^2\theta$, so we would set $x = 4\tan\theta$.

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7 Evaluate $\int \frac{dx}{(9-x^2)^{3/2}}$.

Solution: The goal is to simplify $(9-x^2)^{3/2}$ using the identity $9-9\sin^2\theta = 9\cos^2\theta$, so we set $x = 3\sin\theta$, giving $dx = 3\cos\theta$:

$$\begin{aligned}
\int \frac{dx}{(9-x^2)^{3/2}} &= \int \frac{3\cos\theta}{(9-9\sin^2\theta)^{3/2}}\,d\theta \\
&= \int \frac{3\cos\theta}{3^3\cos^3\theta}\,d\theta \\
&= \frac{1}{9}\int \frac{d\theta}{\cos^2\theta} \\
&= \frac{1}{9}\int \sec^2\theta\,d\theta \\
&= \frac{1}{9}\tan\theta + C.
\end{aligned}$$

Now we would draw a right triangle with $\sin\theta = x/3$ to compute that $\tan\theta = x/\sqrt{9-x^2}$, giving us that

$$\int \frac{dx}{(9-x^2)^{3/2}} = \frac{x}{9\sqrt{9-x^2}} + C.$$

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8 Is the angle between the vectors $\mathbf{a} = \langle 3,-1,2\rangle$ and $\mathbf{b} = \langle 2,2,4\rangle$ acute, obtuse, or right?

Solution: Since $\mathbf{a}\cdot\mathbf{b} = 6-2+8 = 12 > 0$ and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we see that $\cos\theta > 0$, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.

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9 Find the area of the parallelogram whose vertices are $(-1,2,0)$, $(0,4,2)$, $(2,1,-2)$, and $(3,3,0)$.

Solution: Label the points P, Q, R, and S. Then $\overrightarrow{PQ} = \langle 1,2,2\rangle$, $\overrightarrow{PR} = \langle 3,-1,-2\rangle$ and $\overrightarrow{PS} = \langle 4,1,0\rangle$. It follows that the parallelogram is determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, so it's area is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -2,8,-7\rangle| = \sqrt{4+64+49} = \sqrt{117}$.

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10 If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero vectors and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}\times\mathbf{b}|$, what can you say about the relationship between $\mathbf{a}$ and $\mathbf{b}$?

Solution: We are given that $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}\times\mathbf{b}|$, and we know that

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta, \text{ while} \\ |\mathbf{a}\times\mathbf{b}| &= |\mathbf{a}||\mathbf{b}|\sin\theta \end{aligned}$$

It follows that we must have $\cos\theta = \sin\theta$, and the only value of θ which satisfies this is $\theta = \pi/4$, so the two vectors are at a 45° angle to each other.

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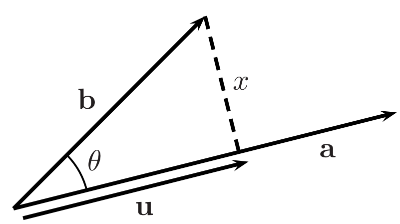
11 Consider the vectors $\mathbf{a} = \langle 4,1\rangle$ and $\mathbf{b} = \langle 2,2\rangle$, shown below. Compute $\cos\theta$, $\mathbf{u}$, and the length x.



Note: you should not leave unevaluated trigonometric functions in your answer.

Solution: As θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we can find it via the dot product:

$$\mathbf{a}\cdot\mathbf{b} = 10 = |\mathbf{a}||\mathbf{b}|\cos\theta = \sqrt{17}\sqrt{8}\cos\theta,$$

so $\cos\theta = \dfrac{10}{\sqrt{17}\sqrt{8}}$.

Now,

$$\mathbf{u} = \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{10}{17}\langle 4,1\rangle = \left\langle \frac{40}{17}, \frac{10}{17}\right\rangle.$$

Finally, using the Pythagorean Theorem,

$$x = \sqrt{|\mathbf{b}|^2 - |\mathbf{u}|^2} = \sqrt{8 - \frac{100}{17}} = \sqrt{\frac{36}{17}} = \frac{6}{\sqrt{16}}.$$

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12 Find the equation of the plane which passes through the point $(2, -3, -1)$ and contains the line

$$x = 3t - 2, \quad y = t + 3, \quad z = 5t - 3 \ .$$

Solution: We need to find two vectors on this plane, so consider the vector from the point $(2, -3, -1)$ to the point $(-2, 3, -3)$ which lies on the given line (just set $t = 0$ in the line equation). This vector is $\langle -4, 6, 2\rangle$, and the direction vector of the given line is $\langle 3, 1, 5\rangle$, so the normal vector to the plane is $\langle -4, 6, 2\rangle \times \langle 3, 1, 5\rangle = \langle 28, 26, -22\rangle$. The equation for the plane is then

$$28(x - 2) + 26(y + 3) - 22(z + 1) = 0.$$

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13 Find the line of intersection of the planes $x + y + z = 12$ and $2x + 3y + z = 2$.

Solution: Since the line of intersection lies on both planes, it must be orthogonal to both normal vectors. Therefore its direction is given by the cross product of the normal vectors:

$$\langle 1, 1, 1\rangle \times \langle 2, 3, 1\rangle = \langle -2, 1, 1\rangle.$$

We also need a point on the line of intersection. To find this, let us set $x = 0$ (other choices work equally well). The equations for the first plane becomes $y + z = 12$, so $z = 12 - y$. Substituting this into the equation for the second plane gives $3y + (12 - y) = 2$, so $y = -5$. Thus the point $(0, -5, 17)$ lies on the line of intersection, so the line is given by

$$x = -2t, \quad y = t - 5, \quad z = t + 17.$$

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14 Compute the position vector for a particle which passes through the origin at time $t = 0$ and has velocity vector

$$\mathbf{v}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}.$$

Solution: The position vector is the antiderivative of the velocity vector, so it is

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = t^2\mathbf{i} - \cos t\mathbf{j} + \sin t\mathbf{k} + \mathbf{C},$$

where $\mathbf{C}$ is a *vector* constant of integration. The problem stated that the particle passes through origin at time $t = 0$, so need $\mathbf{r}(0) = \mathbf{0}$:

$$\mathbf{0} = \mathbf{r}(0) = -\mathbf{j} + \mathbf{C};$$

thus $\mathbf{C} = \mathbf{j}$, so we have that

$$\mathbf{r}(t)t^2\mathbf{i} + (1 - \cos t)\mathbf{j} - \sin t\mathbf{k}.$$

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15 Show that if a particle moves at constant speed, then its velocity and acceleration vectors are orthogonal. Note that this does *not* mean that the velocity is 0! (Hint: consider the derivative of $\mathbf{v} \cdot \mathbf{v}$.)

Solution: Suppose that the particle's speed is C, so $|\mathbf{v}(t)| = C$. Then we have

$$\mathbf{v}(t) \cdot \mathbf{v}(t) = C^2,$$

so taking the derivates of both sides gives

$$\mathbf{v}(t) \cdot \mathbf{v}'(t) + \mathbf{v}'(t) \cdot \mathbf{v}(t) = 0,$$

which implies that $\mathbf{v}(t) \cdot \mathbf{v}'(t) = \mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, as we wanted.

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16 Consider the curve defined by

$$\mathbf{r}(t) = \langle 4\sin ct, 3ct, 4\cos ct \rangle .$$

What value of c makes the arc length of the space curve traced by $\mathbf{r}(t)$, $0 \le t \le 1$, equal to 10?

Solution: The arc length from 0 to 1 of this curve is given by

$$\begin{aligned}
\int_0^1 \text{speed}\, dt &= \int_0^1 \sqrt{16c^2\cos^2 ct + 9c^2 + 16c^2\sin^2 ct}\, dt \\
&= \int_0^1 \sqrt{16c^2 + 9c^2}\, dt \\
&= \int_0^1 5c\, dt \\
&= 5c.
\end{aligned}$$

For this to equal 10, we want $c = 2$.
