NAME : ________________________________

SECTION : (circle one) 12:30–1:35 1:45–2:50

Math 8

February 2, 2010

Midterm 1

$\boxed{\text{SOLUTIONS}}$

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have two hours and you should attempt all problems.

- ***Print*** your name in the space provided and circle your instructor's name.
- Mark your multiple choice answers on the final page of *this booklet*. The multiple choice booklet *will not* be collected.
- Sign the FERPA release on the next page *only if* you wish your exam returned in lecture.
- Calculators or other computing devices are not allowed.
- Use the blank page at the end of the exam for scratch work.
- **Except in the multiple choice section, you must show all work and give a reason (or reasons) for your answer. A CORRECT ANSWER WITH INCORRECT WORK WILL BE CONSIDERED WRONG.**

1. (10) Compute the Taylor polynomial of degree 4 centered at $x = 0$ for the function $f(x) = \cos 2x$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$\boxed{T_4(x) = 1 - \frac{4x^2}{2} + \frac{16x^3}{4!}}$$

(b) Using the Remainder Theorem, for what values of x is this Taylor polynomial guaranteed to be within $1/120$ of the true value of $f(x)$?

$f(x) = \cos 2x$

$f'(x) = -2 \sin 2x$

$f''(x) = -4 \cos 2x$

$f'''(x) = 8 \sin 2x$

$f^{(4)}(x) = +16 \cos 2x$

$f^{(5)}(x) = -32 \cos 2x$

$$R_4(x) = \frac{f^{(5)}(c)}{5!} |x|^5$$

$$|R_4(x)| \leq \frac{32}{120} |x|^5$$

$$\frac{32}{120} |x|^5 \leq \frac{1}{120}$$

$$|x|^5 \leq \frac{1}{32}$$

$$\boxed{|x| \leq \frac{1}{2}}$$

2. (10) What is the sum of the series $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$?

Geometric series.

First term = 2.

Ratio = $-\frac{1}{2}$.

$$\text{Sum} = \frac{2}{1-(-\frac{1}{2})} = \frac{2}{3/2} = \boxed{\frac{4}{3}.}$$

(b) Could you make this series converge to a different sum by rearranging it? Why or why not?

$\sum_{n=0}^{\infty} \left|2\left(-\frac{1}{2}\right)^n\right|$ converges, so the series is $\boxed{\text{absolutely convergent.}}$

All rearrangements converge to the same sum.

3. (10) Suppose that $p > 1$. Does the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p}$$

converge absolutely, converge conditionally, or diverge? You should mention any tests you apply, and make sure that the series satisfies the conditions of those tests.

Integral Test.

$$\int_3^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x(\ln x)^p}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_{x=3}^{x=b} \frac{1}{u^p}\,du$$

$$= \lim_{b\to\infty} \left.\frac{u^{-p+1}}{-p+1}\right|_{x=3}^{\infty}$$

$$= \lim_{b\to\infty} \frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 3)^{-p+1}}{-p+1}$$

Since $p>1$, $-p+1<0$, so $(\ln b)^{-p+1} \to 0$ as $b\to\infty$.

The series $\boxed{\text{converges.}}$

4. (10) Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{6^n \sqrt{n}}.$$

Ratio Test for absolute convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)}{6}\frac{\sqrt{n+1}}{\sqrt{n}}\right| \to \left|\frac{x-2}{6}\right| \text{ as } n \to \infty.$$

$$\left|\frac{x-2}{6}\right| < 1$$

$$|x-2| < 6$$

$$-6 < x-2 < 6$$

$$\boxed{-4 < x < 8}$$

Now check endpoints.

At $x=-4$: $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by AST.

At $x=8$: $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series.

$\boxed{\text{Interval of Convergence: } [-4, 8).}$

5. (10) Derive a power series centered at $x = 0$ for the function

$$f(x) = \ln(1 + x).$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C.$$

Plug in $x=0$ to solve for C:

$$\ln(1+0) = 0 = C.$$

So: $\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}.$

(b) Find a power series centered at $x = 0$ for the function

$$f(x) = 2x^3 \ln(1 + 2x^2).$$

(You may use your answer above.)

$$f(x) = 2x^3 \ln(1+2x^2) = 2x^3 \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1} x^{2n+2}}{n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} 2(-1)^n \frac{2^{n+1} x^{2n+5}}{n+1}}.$$

6. (10) Express $\int \sin x^2\,dx$ as a power series centered at $x = 0$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$\int \sin x^2\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx$$

$$\boxed{= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C.}$$

7. (4) Suppose you know that $\lim_{n\to\infty} a_n = 0$. What can you conclude about $\sum_{n=1}^{\infty} a_n$?

A. it diverges

B. is converges

C. if the terms are positive, it converges

D. if the terms are positive and decreasing, it converges

E. nothing

$\to$ consider $\sum \frac{1}{n}$.

8. (4) Which of the following statements are true about the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$?

I	The series diverges by the Ratio Test.
II	The series diverges by comparison to the harmonic series.
III	The series diverges by the Test for Divergence.

A. None

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II, and III

Ratio Test inconclusive — I is false.

$\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$ — II is true.

$\frac{\ln n}{n} \to 0$ — III is false.

9. (4) What is the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n n!}$?

Ratio Test for absolute convergence:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x+3)}{2(n+1)} \right| \to 0 \text{ as } n \to \infty.$$

Series converges everywhere.

A. $(-5, -1]$

B. $[-5, 1]$

C. $[-1, 5]$

D. $[-1, 5)$

E. $(-\infty, \infty)$

10. (4) Which of the following series converge (either absolutely or conditionally)?

I	II	III
$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^3+2}}{8n+11}$	$\sum_{n=1}^{\infty} \frac{\sin^6 n}{\pi^n}$	$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$

I diverges by the Test for Divergence.

II converges by comparison to $\sum \left(\frac{1}{\pi}\right)^n$.

III diverges by Integral Test.

A. None

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II, and III

11. (4) Suppose that the series

$$f(x) = 3 + 2(x-1) + \frac{(x-1)^2}{7} - \frac{(x-1)^3}{3} + \frac{(x-1)^4}{12} + \cdots$$

converges for all values of x. What is $f'''(1)$?

A. 3

B. $1/3$

C. $-1/3$

D. 2

E. -2

$f'''(x) = -2 + 2(x-1) + \cdots$

$f'''(1) = -2$

12. (4) Which of the following series converge *absolutely*?

I	II	III
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$	$\sum_{n=1}^{\infty} \left(\frac{-5}{6}\right)^n$	$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

A. None

B. *I* only

C. *II* only

D. *III* only

E. *I* and *II* only

F. *I* and *III* only

G. *II* and *III* only

H. *I*, *II*, and *III*

I converges absolutely – p-test

II converges absolutely – geometric

III only converges conditionally

(Integral Test shows $\sum \frac{1}{n \ln n}$ diverges)

13. (4) What is the Taylor series centered at $x = 0$ for the function

$$f(x) = -xe^{-x^2}?$$

A. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n!}$

B. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}$

C. $\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}$

D. $\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{(2n)!}$

E. $\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$-xe^{-x^2} = -x\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n!}$$

14. (4) Which of the following statements are true about the sequence $\{1/n\}$?

I	The sequence converges.
II	The sequence is monotone.
III	The sequence is bounded.

A. None

B. *I* only

C. *II* only

D. *III* only

E. *I* and *II* only

F. *I* and *III* only

G. *II* and *III* only

H. *I*, *II*, and *III*

I: $\frac{1}{n} \to 0$

II: Yes, it is decreasing

III: Yes, $0 \leq \frac{1}{n} \leq 1$ for all n.

15. (4) Two 50% marksmen decide to fight in a duel in which they exchange shots until one of them is hit. What is the chance that the first shooter wins?

A. 1/3
B. 1/2
C. 2/3
D. 3/4
E. 1

Chance for first shooter:

$$\underset{\text{hit}}{\frac{1}{2}} + \underbrace{\frac{1}{2}\frac{1}{2}}_{\text{misses}}\underset{\text{hit}}{\frac{1}{2}} + \underbrace{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}_{\text{misses}}\underset{\text{hit}}{\frac{1}{2}} + \cdots$$

First term 1/2
Ratio 1/4

$$\text{Sum} = \frac{1/2}{1-1/4} = \frac{2}{3}.$$

16. (4) Suppose that $\sum_{n=0}^{\infty} c_n 4^n$ converges and $\sum_{n=0}^{\infty} c_n(-5)^n$ diverges. Which of the following statements is correct?

I	The series $\sum_{n=0}^{\infty} c_n 3^n$ converges.
II	The series $\sum_{n=0}^{\infty} c_n(-2)^n$ converges.
III	The series $\sum_{n=0}^{\infty} c_n 6^n$ diverges.

A. None
B. I only
C. II only
D. III only
E. I and II only
F. I and III only
G. II and III only
H. I, II, and III

Center is 0.
Radius is between 4 & 5.

I converges
II converges
III diverges