2.7. THE ALTERNATING SERIES TEST

We have focused almost exclusively on series with positive terms up to this point. In this short section we begin to delve into series with both positive and negative terms, presenting a test which works for many series whose terms alternate in sign.

The Alternating Series Test. Suppose that the sequence $\{b_n\}$ satisfies the three conditions:

- $b_n \geqslant 0$ for sufficiently large n,
- $b_{n+1} \leqslant b_n$ for sufficiently large n (i.e., $\{b_n\}$ is monotonically decreasing), and
- $b_n \to 0$ as $n \to \infty$.

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

While we have stated the test with $(-1)^{n+1}$, it of course applies if the terms involve $(-1)^n$ instead (or $\cos n\pi$, since this is just a convoluted way to write $(-1)^n$). Also, notice that the Alternating Series Test can *not* be used to show that a series diverges (see Example 2).

Proof of the Alternating Series Test. Assume that the sequence $\{b_n\}$ is positive and decreasing for all n, and that it has limit 0. By the Tail Observation of Section 2.2, if we can prove that these series converge, the full Alternating Series Test will follow.

Let s_n denote the nth partial sum of this series. We have

$$s_{2n} = (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2n-1} - b_{2n}).$$

Because $\{b_n\}$ is monotonically decreasing, $b_{2n-1} - b_{2n} \geqslant 0$ for all n, so this shows that s_{2n} is monotonically increasing. We can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n},$$

so since $b_{2n-2} - b_{2n-1} \geqslant 0$, $s_{2n} < b_1$. Thus the sequence $\{s_{2n}\}$ has a limit by the Monotone Convergence Theorem. Let $L = \lim_{n\to\infty} s_{2n}$. Now we consider the odd partial sums: $s_{2n+1} = s_{2n} + b_{2n+1}$, so

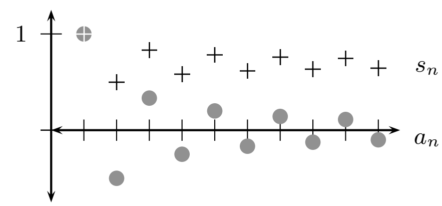
$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_n = L,$$

because $b_n \to 0$ by our hypotheses. Since both the even and odd partial sums converge to the same value, the sum of the series exists. □

Example 1 (The Alternating Harmonic Series, again). Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges using the Alternating Series Test.



Solution. The sequence $\{1/n\}$ is positive, monotonically decreasing, and has limit 0, so the alternating harmonic series converges by the Alternating Series Test. ●

Example 2. Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{3n+4}$ converge or diverge?

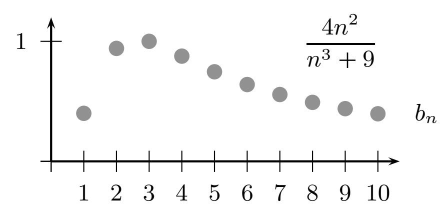
Solution. This series does alternate in sign, and $(2n+3)/(3n+4)$ is decreasing, but

$$(2n+3)/(3n+4) \to 2/3 \neq 0,$$

so the series diverges by the Test for Divergence. ●

Note that in the solution of Example 2, we did not appeal to the Alternating Series Test, but instead used the Test for Divergence. The Alternating Series Test *never* shows that series diverge.

Example 3. Show that the series $\sum_{n=1}^{\infty} (-1)^n \frac{4n^2}{n^3+9}$ converges.



Solution. This series alternates in sign, and $(4n^2)/(n^3+9) \to 0$, but it is not immediately obvious that the sequence $\{(4n^2)/(n^3+9)\}$ is decreasing. Indeed, its first three terms are increasing, as indicated in the plot. Of course, we only need the sequence to be monotonically decreasing for large n. To check this condition, we take a derivative:

$$\frac{d}{dx}\frac{4x^2}{x^3+9} = \frac{(x^3+9)(8x)-(4x^2)(3x^2)}{(x^3+9)^2} = \frac{-4x^4+72x}{(x^3+9)^2}.$$

This fraction is negative for large x, so the sequence $\{(4n^2)/(n^3+9)\}$ is decreasing for large n. Therefore the series converges by the Alternating Series Test. ●

The proof of the Alternating Series Test implies the following very simple bound on remainders of these series.

The Alternating Series Remainder Estimates. Suppose that the sequence $\{b_n\}$ satisfies the three conditions of the Alternating Series Test:

- $b_n \geqslant 0$,
- $b_{n+1} \leqslant b_n$, and
- $b_n \to 0$ as $n \to \infty$

for all $n \geqslant N$. Then if $n \geqslant N$, the error in the nth partial sum of $\sum(-1)^{n+1}b_n$ is bounded by b_{n+1}:

$$\left| s_n - \sum_{n=1}^{\infty}(-1)^{n+1}b_n \right| \leqslant b_{n+1}.$$

Example 4. How many terms of the alternating series must we add to approximate the true sum with error less than $1/10000$?

Solution. Since the alternating harmonic series $\sum(-1)^{n+1}/n$ satisfies the conditions of the Alternating Series Test for all $n \geqslant 0$, the Remainder Estimates show that

$$\left| s_n - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \right| \leqslant \frac{1}{n+1}.$$

Therefore, if we want the error to be less than $1/10000$, we need

$$\text{Error} \leqslant \frac{1}{n+1} < \frac{1}{10000},$$

so we need $n > 9999$, or in other words, $n \geqslant 10000$. ●

EXERCISES FOR SECTION 2.7

In Exercises 1–12, determine if the given series converge or diverge.

$\boxed{\textbf{1.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+7}$

$\boxed{\textbf{2.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n!}$

$\boxed{\textbf{3.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

$\boxed{\textbf{4.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$

$\boxed{\textbf{5.}}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n-1}{n}$

$\boxed{\textbf{6.}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$

$\boxed{\textbf{7.}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

$\boxed{\textbf{8.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^2}$

$\boxed{\textbf{9.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^{3n}}$

$\boxed{\textbf{10.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$

$\boxed{\textbf{11.}}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{n^3+1}$

$\boxed{\textbf{12.}}$ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2}{n}\right)^{3n}$

For Exercises 13–16, first determine if the given series satisfies the conditions of the Alternating Series Test. Then, if the series does satisfy the conditions, decide how many terms need to be added in order to approximate the sum to within $1/1000$.

$\boxed{\textbf{13.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^6+1}$

$\boxed{\textbf{14.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

$\boxed{\textbf{15.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}$

$\boxed{\textbf{16.}}$ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\right)^2$

Exercises 17–19 verify that the hypotheses of the Alternating Series Test are all necessary, in the sense that if any of them is removed, then the statement becomes false.

17. Construct a sequence $\{b_n\}$ which is monotonically decreasing with limit 0 such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't be positive.)

18. Construct a sequence $\{b_n\}$ which is positive and monotonically decreasing such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't have limit 0.)

19. Construct a sequence $\{b_n\}$ which is positive with limit 0 such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't be monotonically decreasing.)

Dirichlet's Test, due to Johann Peter Gustav Lejeune Dirichlet (1805–1859), is a strengthening of the Alternating Series Test (as shown in Exercise 23).

Dirichlet's Test. If $\{b_n\}$ is a positive, eventually monotonically decreasing sequence with limit 0 and the partial sums of the series $\sum a_n$ are bounded, then $\sum a_n b_n$ converges.

Exercises 20–22 ask you to develop the proof of this theorem, while Exercises 23–27 ask you to apply the test.

◆ **20.** Let $s_n = a_1 + a_2 + \cdots + a_n$. Use the fact that $s_n - s_{n-1} = a_n$ to prove

$$\sum_{n=m+1}^{\infty} a_n b_n = -s_m b_{m+1} + \sum_{n=m+1}^{\infty} s_n(b_n - b_{n+1}).$$

(This formula is often referred to as *summation by parts*.)

◆ **21.** Let $\{a_n\}$ and $\{b_n\}$ be sequences satisfying the hypotheses of Dirichlet's Test. Use Exercise 20 to show that if the partial sums of the sequence $\{a_n\}$ are at most M then

$$\left|\sum_{n=m+1}^{\infty} a_n b_n\right| \leqslant 2M\big|b_{m+1}\big|.$$

◆ **22.** Use Exercise 21 to prove Dirichlet's Test.

◆ **23.** Show that Dirichlet's Test implies the Alternating Sign Test.

24. Suppose that $\{a_n\} = \{-2, 4, 1, -3, -2, 4, 1, -3, \dots\}$ and that $b_n = 1/n$. Does $\sum a_n b_n$ diverge, converge absolutely, or converge conditionally?

◆ **25.** Use the angle addition identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

to derive the identity

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta)\cos(\alpha + \beta).$$

◆ **26.** Use the identity derived in Exercise 25 to show that

$$2\left(\sin{}^{\pi}/_{4}\right)\sum_{n=1}^{m}\sin n = \sum_{n=1}^{m}\left(\cos\left(n - {}^{\pi}/_{4}\right) - \cos\left(n + {}^{\pi}/_{4}\right)\right),$$

then show that this is equal to $\cos{}^{\pi}/_{4} - \cos\left(m + {}^{\pi}/_{4}\right)$.

◆ **27.** Use Exercise 26 to show that the partial sums of $\sum \sin n$ are bounded, and then conclude from Dirichlet's Test that the series

$$\sum_{n=1}^{\infty}\frac{\sin n}{n}$$

converges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.7

1. Converges by the Alternating Series Test
3. Converges by the Alternating Series Test
5. Diverges by the Test for Divergence:
$$\frac{n-1}{n} \to 1 \neq 0$$
as $n \to \infty$.
7. Converges by the Alternating Series Test
9. Converges by the Alternating Series Test
11. Converges by the Alternating Series Test. To see that $\{b_n\}$ is decreasing for sufficiently large n, take a derivative.
13. Alternating Series Test not applicable.
15. The Alternating Series Test is applicable. Using $n = 4$ will work to approximate the sum to within $1/1000$, because
$$\frac{1}{(5!)^2} = \frac{1}{14400} < \frac{1}{1000}.$$

2.8. ABSOLUTE VS. CONDITIONAL CONVERGENCE

We are now ready to examine the strange behavior of the alternating harmonic series we first observed in Section 2.2. Remember that we showed that the alternating harmonic series converged and then we went on to bound its sum. For a lower bound, we grouped the terms in pairs, observing that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots\\ &\geqslant 1-\frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

While for an upper bound, we group the terms in different pairs, showing that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\ &= 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)-\cdots\\ &\leqslant 1-\frac{1}{2}+\frac{1}{3}\\ &= \frac{5}{6}=0.8333\ldots.\end{aligned}$$

(In fact, that true sum is $\ln 2\approx 0.69315$, see Exercises 46 and 47 of Section 2.4 or Exercise 24 of Section 3.2.)

Then we showed in Example 5 of Section 2.2 that by rearranging the terms of this series, we could get it to converge to a different sum:

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)\geqslant\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)=\frac{389}{420}>\frac{9}{10}.$$

Our first order of business in this section is to explore this phenomenon:

When are we allowed to rearrange the terms of a series without changing the sum?

We begin by looking at series with positive terms. If $\sum a_n$ is a convergent series with positive terms, are we allowed to rearrange the terms without changing the sum? Suppose $\sum b_n$ is such a rearrangement, and consider the partial sums of each series,

$$\begin{aligned}s_n &= a_1+a_2+a_3+\cdots,\\ t_n &= b_1+b_2+b_3+\cdots.\end{aligned}$$

We would like to figure out if $\sum a_n = \sum b_n$ which, by the very definition of series summation, is equivalent to $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. Because $\{a_n\}$ and $\{b_n\}$ are positive sequences, the sequences of partial sums $\{s_n\}$ and $\{t_n\}$ are both increasing. Now consider any value of n. Since the sequence $\{b_n\}$ is a rearrangement of the sequence $\{a_n\}$, there must be some number N so that each of the terms

$$a_1, a_2, \ldots, a_n$$

occurs in the list

$$b_1, b_2, \ldots, b_N.$$

Since all the terms are positive, for this value of N, we have

$$s_n = a_1 + a_2 + \cdots + a_n \leqslant b_1 + b_2 + \cdots + b_N = t_N.$$

This shows that every partial sum of $\sum a_n$ is less than or equal to some partial sum of $\sum b_n$. Of course, the same argument works with the roles of a_n and b_n interchanged, so every partial sum of $\sum b_n$ is less than or equal to some partial some of $\sum a_n$. This implies that the two series converge to the same value.

So we have made some progress: convergent series with positive terms can be rearranged without affecting their sums, but rearranging the alternating harmonic series can affect its sum. What is the difference between these two examples?

Intuitively, there are two different ways for a series to converge. First, the terms could just be really small. Indeed, this is the only way that a series with positive terms can converge. But then there is a second way, illustrated by the alternating harmonic series: the terms could cancel each other out. Our next definition attempts to make precise the notion of series that converge "because their terms are really small."

Absolute Convergence. The series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ converges.

The first thing we should verify is that absolutely convergent series actually, well, converge. Our next theorem says even more: rearrangements don't affect the sum of an absolutely convergent series.

The Absolute Convergence Theorem. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges. Moreover, every rearrangement of $\sum a_n$ converges to the same sum.

This first part of this theorem — that absolutely convergent series converge — follows from the Comparison Test and some basic facts about series, see Exercises 25–26. The second part is more complicated, and we omit its proof.

While we have defined absolute convergence in order to investigate rearranging series, this notion is very useful on its own. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Even though this series is very much like $\sum 1/n^2$, it is *not* a p-series, so we can't apply the p-series Test to it. Similarly, we can't apply the Integral Test or the Comparison Test, because those tests require series to have positive terms. However, it is easy to see that this series is absolutely convergent, from which it follows that the series converges by the Absolute Convergence Theorem:

Example 1. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

Solution. The series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2}$ converges absolutely by the Absolute Convergence Theorem. ●

Our next example is another stereotypical use of the Absolute Convergence Theorem. In general when trigonometric functions appear in a series, we need to test for absolute convergence and then make a comparison.

Example 2. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$ converges absolutely.

Solution. First we take the absolute values of the terms,

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{\sqrt{n^3+1}}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}}.$$

We may use any test we like on this series (although some, like the Ratio Test in this example, might not tell us anything). Because $|\sin n| \leqslant 1$, $\sqrt{n^3+1} \geqslant \sqrt{n^3} = n^{3/2}$, and the terms of this series are positive, we can compare it:

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}}$ is convergent by comparison to a convergent p-series, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$ is absolutely convergent by the Absolute Convergence Theorem. ●

We've identified a special type of convergent series, the absolutely convergent series. But what about the others? Intuitively, these are the series which converge only because their terms happen to cancel each other out. These series are called conditionally convergent.

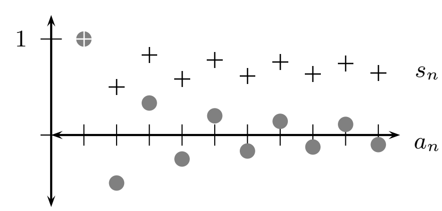
Conditional Convergence. The series $\sum a_n$ is said to *converge conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

If you want to show that the series $\sum a_n$ is conditionally convergent, it is important to note that this requires two steps. First you must show that $\sum a_n$ converges, and second, you must show that $\sum a_n$ is not absolutely convergent (in other words, that $\sum |a_n|$ diverges). Our first example of a conditionally convergent series should not come as a surprise.

Example 3 (The Alternating Harmonic Series, last time). Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is conditionally convergent.



Solution. The alternating harmonic series converges by the Alternating Series Test because the sequence $\{1/n\}$ is monotonically decreasing, positive, and has limit 0.

The alternating harmonic series does not converge absolutely because

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

(the harmonic series) diverges. Therefore the alternating harmonic series is conditionally convergent. ●

Example 4. Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4n^2}{n^3+9}$ converges conditionally.

Solution. We saw in Example 3 of the previous section that this series converges, so we only need to show that it does not converge absolutely. To test for absolute convergence, we take the absolute value:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{4n^2}{n^3+9} \right| = \sum_{n=1}^{\infty} \frac{4n^2}{n^3+9}.$$

There are at least two different ways to show that this series diverges.

With the Integral Test: We must evaluate the integral

$$\int_1^{\infty} \frac{4x^2}{x^3+9}\,dx = \lim_{b\to\infty} \int_1^b \frac{4x^2}{x^3+9}\,dx.$$

Setting $u = x^3 + 9$ gives $du = 3x^2\,dx$, so $dx = du/3x^2$. Making these substitutions leaves us with

$$\lim_{b\to\infty} \int_{x=1}^{x=b} \frac{4}{3u}\,du = \lim_{b\to\infty} \frac{4}{3} \ln u \bigg|_{x=1}^{x=b} = \lim_{b\to\infty} \frac{4}{3} \ln\left(b^3+9\right) - \frac{4}{3}\ln(10) = \infty,$$

so the series diverges by the Integral Test.

With the Comparison Test: Here we can use the bound

$$\frac{4x^2}{x^3+9} \geqslant \frac{4x^2}{x^3+9x^3} = \frac{4}{10x}$$

to see that the series diverges by comparison to $4/10 \sum 1/n$. ●

We've seen one example of how by rearranging the terms of the alternating harmonic series we can change its sum. What if we wanted to rearrange the series to make it sum to a specific number? Would that be possible? Yes! We begin with a specific example, and then discuss how to generalize this example.

Example 5. Rearrange the terms of the alternating harmonic series to get a series which converges to 1.

Solution. The positive terms of this series are

$$1 + 1/3 + 1/5 + 1/7 + 1/9 + \cdots$$

while the negative terms are

$$-1/2 - 1/4 - 1/6 - 1/8 - 1/10 - \cdots.$$

Note that both of these series diverge. By our Tail Observation of Section 2.2, this means that all tails of these series diverge as well.

Now, how are we going to rearrange the series to make it sum to 1? First, we make the series sum to *more* than 1:

$$1 + 1/3 \quad = \quad 1.3333\ldots \quad > \quad 1.$$

Next we use negative terms to make the series sum to *less* than 1:

$$1 + 1/3 - 1/2 \quad = \quad 0.8333\ldots \quad < \quad 1.$$

Then we use as many of the positive terms that we haven't used yet to make the series sum to more than 1 again:

$$1 + 1/3 - 1/2 + 1/5 \quad = \quad 1.0333\ldots \quad > \quad 1,$$

and then use negative terms to make it sum to less than 1:

$$1 + 1/3 - 1/2 + 1/5 - 1/4 \quad = \quad 0.7833\ldots \quad < \quad 1.$$

In doing so we obtain with the rearrangement

$$\underbrace{\overbrace{\underbrace{\overbrace{\underbrace{\overbrace{1 + 1/3}^{>1} - 1/2}_{<1} + 1/5}^{>1} - 1/4}_{<1} + 1/7 + 1/9}^{>1} - 1/6}_{<1} + \cdots\ ,$$

but does this rearrangement really sum to 1? Mightn't we get stuck at some point and not be able to continue the construction?

We certainly won't get stuck under 1. No matter how many of the positive terms we have used up to that point, the positive terms that we have remaining will sum to ∞ (they are a tail of the divergent series $1 + 1/3 + 1/5 + \cdots$). Similarly, we can't get stuck over 1. Therefore, we will be able to create partial sums which are alternatively greater than 1 and less than 1, but will they converge to 1? This follows because the terms we are using are getting smaller. If we add the term $1/93$ to get a partial sum over 1, that means that our previous partial sum was *under* 1, which means that the new partial sum is within $1/93$ of 1. As we use up the larger terms of the series, we will have no choice but to get closer and closer to 1. Therefore this construction (if we carried it out forever) would indeed yield a sum of 1. ●

Now we know we can rearrange the alternating harmonic series to sum to 1, but what was so special about 1? Absolutely nothing, in fact. If you replace the number 1 in the previous argument with any other number S, everything works just fine. Now, what was so special about the alternating harmonic series? First, we needed that the positive terms formed a divergent series and that the negative terms formed a divergent series (so that our partial sums wouldn't get stuck under or above 1). This fact is actually true for all conditionally convergent series though (why?). Then we needed that the terms get increasingly small, to prove that the limit of the partial sums was really 1. But if the terms didn't get close to 0, then the series would diverge by the Test for Divergence, so this is true for all conditionally convergent series as well.

We have just sketched the proof of a famous theorem of Bernhard Riemann (1826–1866).

> **Reimann's Rearrangement Theorem.** If $\sum a_n$ is a conditionally convergent series and S is any real number, then there is a rearrangment of $\sum s_n$ which converges to S.

We conclude with a more formal proof.

Proof of Reimann's Rearrangement Theorem We begin by dividing the terms of the sequence $\{a_n\}$ into two groups. Let $\{b_n\}$ denote the sequence which contains the positive terms of $\{a_n\}$ and $\{c_n\}$ denote the sequence which contains the negative terms of $\{a_n\}$.

Clearly $\sum |a_n| = \sum b_n - \sum c_n$, so since $\sum a_n$ is not absolutely convergent, at least one of $\sum b_n$ or $\sum c_n$ must diverge. But $\sum a_n$ is conditionally convergent, so if $\sum b_n$ diverges (to ∞), $\sum c_n$ must also diverge (to $-\infty$), and vice versa. Therefore both $\sum b_n$ and $\sum c_n$ diverge, to ∞ and $-\infty$, respectively.

Suppose that a target sum S is given. Choose N_1 to be the minimal integer such that

$$b_1 + \cdots + b_{N_1} > S$$

(note that if S is negative, then N_1 will be 0). We can be certain that N_1 exists because $\sum b_n$ diverges to ∞. Note that, because $b_1 + \cdots + b_{N_1 - 1} < S$, $b_1 + \cdots + b_{N_1}$ is within b_{N_1} of S. Next choose M_1 minimal so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) < S.$$

Again, M_1 must exist because $\sum c_n$ diverges to $-\infty$. Note that any partial sum of the form $(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_n)$ where $n \leqslant M_1$ must be within b_{N_1} of S. Next choose N_2 so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) + (b_{N_1+1} + \cdots + b_{N_2}) > S.$$

Next we choose M_2 so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) + (b_{N_1+1} + \cdots + b_{N_2}) - (c_{M_1+1} + \cdots + c_{M_2}) < S.$$

Continuing in this manner, define $N_3, M_3, \ldots$. At each stage, our partial sums will be within b_{N_i} or c_{M_i} of S for some i, and so since $b_n \to \infty$ and $c_n \to \infty$ (why?) we obtain a rearrangement that sums to S, as desired. $\square$

EXERCISES FOR SECTION 2.8

For Exercises 1–12, determine if the given series converge absolutely, converge conditionally, or diverge. Note that these exercises may require the use of all the tests we have learned thus far.

$\boxed{\textbf{1.}}$ $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

$\boxed{\textbf{2.}}$ $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

$\boxed{\textbf{3.}}$ $\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$

$\boxed{\textbf{4.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

$\boxed{\textbf{5.}}$ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$\boxed{\textbf{6.}}$ $\sum_{n=1}^{\infty} \frac{1}{(n)(\ln n)(\ln \ln n)}$

$\boxed{\textbf{7.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

$\boxed{\textbf{8.}}$ $\sum_{n=1}^{\infty} (-1)^n \cos(1/n)$

$\boxed{\textbf{9.}}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

$\boxed{\textbf{10.}}$ $\sum_{n=1}^{\infty} \frac{\cos n^4 + \sin n^5}{n^2}$

$\boxed{\textbf{11.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{e^n}$

$\boxed{\textbf{12.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln \ln \ln \ln \ln n}$

Determine if the series in Exercises 13–16 converge at $x = -1$ and at $x = 5$.

$\boxed{\textbf{13.}}$ $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

$\boxed{\textbf{14.}}$ $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \sqrt{n}}$

$\boxed{\textbf{15.}}$ $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n^2}$

$\boxed{\textbf{16.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n \sqrt{n}}$

Determine whether the series in Exercises 17–20 converge at $x = -2$ and at $x = 4$.

17. $\sum_{n=0}^{\infty} \frac{n^2}{\sqrt{n^9+5}} \left(\frac{x-1}{3}\right)^n$

18. $\sum_{n=0}^{\infty} (x-1)^2 \frac{n^3}{3^n n^7 + 2n}$

19. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{2n+3^n} (x-1)^n$

20. $\sum_{n=0}^{\infty} \frac{\sin n}{n^3} \left(\frac{x-1}{3}\right)^n$

Determine whether the series in Exercises 21–24 converge at $x = 0$ and at $x = 4$.

21. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n n^{3/2}}$

22. $\sum_{n=1}^{\infty} \frac{1}{n \ln n} \left(\frac{x-2}{2}\right)^n$

23. $\sum_{n=0}^{\infty} \frac{(2-x)^n}{n2^n + 2^n}$

24. $\sum_{n=0}^{\infty} \frac{(x-2)^n \sin n}{2^n}$

Exercises 25 and 26 prove the first part of the Absolute Convergence Theorem: absolutely convergent series converge.

25. Verify the inequality

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|$$

and use this to prove that if $\sum a_n$ is absolutely convergent, then the series $\sum (a_n + |a_n|)$ converges.

26. Use the conclusion of Exercise 25 and Exercise 27 from Section 2.2 to prove that all absolutely convergent series converge.

ANSWERS TO SELECTED EXERCISES, SECTION 2.8

1. Absolutely convergent (use a comparison on the absolute values)
3. Absolutely convergent (use the Ratio Test on the absolute values)
5. Absolutely convergent (use the Integral Test)
7. Conditionally convergent (use the Alternating Series Test, and then use a comparison on the absolute values)
9. Conditionally convergent (use the Alternating Series Test, and then use a comparison on the absolute values)
11. Absolutely convergent (use the Ratio Test on the absolute values)
13. Diverges at both $x = -1$ and $x = 5$
15. Converges at $x = -1$, diverges at $x = 5$