2.6. THE RATIO TEST

We now know how to handle series which we can integrate (the Integral Test), and series which are similar to geometric or p-series (the Comparison Test), but of course there are a great many series for which these two tests are not ideally suited, for example, the series

$$\sum_{n=1}^{\infty} \frac{4^n}{n!}.$$

Integrating the terms of this series would be difficult, especially since the first step would be to find a continuous function which agrees with $n!$ (this can be done, but the solution is not easy). We could try a comparison, but again, the solution is not particular obvious (indeed, those readers who solved Exercise 37 of the last section should feel proud). Instead, the simplest approach to such a series is the following test due to Jean le Rond d'Alembert (1717–1783).

The Ratio Test. Suppose that $\sum a_n$ is a series with positive terms and let $L = \lim_{n\to\infty} a_{n+1}/a_n$.

- If $L < 1$ then $\sum a_n$ converges.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ or the limit does not exist then the Ratio Test is inconclusive.

You shold think of the Ratio Test as a generalization of the Geometric Series Test. For example, if $\{a_n\} = \{ar^n\}$ is a geometric sequence then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

and we know these series converge if and only if $|r| < 1$. (Note that we will only consider positive series here; we deal with mixed series in the next section.) In fact, the proof of the Ratio Test is little more than an application on the Comparison Test.

Proof. If $L > 1$ then the sequence $\{a_n\}$ is increasing (for sufficiently large n), and therefore the series diverges by the Test for Divergence.

Now suppose that $L < 1$. Choose a number r sandwiched between L and 1: $L < r < 1$. Because $a_{n+1}/a_n \to L$, there is some integer N such that

$$0 \leqslant a_{n+1}/a_n \leqslant r$$

for all $n \geqslant N$. Set $a = a_N$. Then we have

$$a_{N+1} \leqslant r a_N = ar,$$

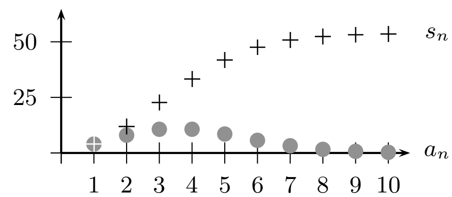
and

$$a_{N+2} \leqslant r a_{N+1} < ar^2,$$

and in general, $a_{N+k} \leqslant ar^k$. Therefore for sufficiently large n (namely, $n \geqslant N$), the terms of the series $\sum a_n$ are bounded by the terms of a convergent geometric series (since $0 < r < 1$), and so $\sum a_n$ converges by the Comparison Test. □

Since the Ratio Test involves a ratio, it is particularly effective when series contain factorials, as our first example does.

Example 1. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$ converge or diverge?



Solution. First we compute L:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{4}{n+1} = 0.$$

Since $L = 0$, this series converges by the Ratio Test. ●

It is important to note that the Ratio Test is *always* inconclusive for series of the form $\sum \frac{\text{polynomial}}{\text{polynomial}}$. As an example, we consider the harmonic series and $\sum 1/n^2$.

Example 2. Show that the Ratio Test is inconclusive for $\sum 1/n$ and $\sum 1/n^2$.

Solution. For the harmonic series, we have

$$L = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1}.$$

In order to evaluate this limit, remember that we factor out the highest order term:

$$L = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{1}{1+\frac{1}{n}} = 1,$$

so the test is inconclusive.

The series $\sum 1/n^2$ fails similarly:

$$L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2},$$

and again we factor out the highest order term, leaving

$$L = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{1}{(1+\frac{1}{n})^2} = 1,$$

so neither series can be handled by the Ratio Test. ●

As Example 2 demonstrates, knowing that $a_{n+1}/a_n < 1$ is *not* enough to conclude that the sequence converges; we must know that the *limit* of this ratio is less than 1.

Example 3. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n4^{2n+1}}$ converge or diverge?

Solution. The ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{10^{n+1}}{(n+1)4^{2n+3}}}{\dfrac{10^n}{n4^{2n+1}}} = \frac{10n}{4^2(n+1)} \to \frac{10}{16}$$

as $n \to \infty$. Since this limit is less than 1, we can conclude that the series converges by the Ratio Test. ●

The last example could also be handled by the Comparison Test, since

$$\frac{10^n}{n4^{2n+1}} \leqslant \frac{10^n}{4^{2n+1}} = \frac{1}{4}\left(\frac{10}{16}\right)^n,$$

so the series converges by comparison with a convergent geometric series. However, what if we moved the n from the denominator to the numerator:

$$\sum_{n=1}^{\infty} \frac{n10^n}{4^{2n+1}}?$$

Now the inequality in the comparison goes the wrong way, making the Comparison Test much harder to use. On the other hand, the limit in the Ratio Test is unchanged (you should check this for yourself). In general, it is usually a good idea to try the Ratio Test on all series with exponentials (like 10^n) or factorials.

Example 4. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}$ converge or diverge?

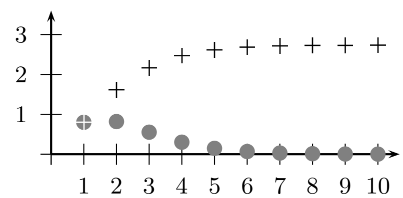
Solution. Here the ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(2n+2)!}{2^{n+1}(n+1)!}}{\dfrac{(2n)!}{2^n n!}} = \frac{(2n+2)(2n+1)}{2(n+1)} = 2n+1 \to \infty$$

as $n \to \infty$. Since this limit is greater than 1 (or any other number, for that matter), the series diverges by the Ratio Test. ●

Our last example could be done using the Comparison Test (how?), but it is (probably) easier to use the Ratio Test.

Example 5. Does the series $\sum_{n=1}^{\infty} \frac{n^2+2n+1}{3^n+2}$ converge or diverge?



Solution. In this case the ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2+2(n+1)+1}{3^{n+1}+2}}{\dfrac{n^2+2n+1}{3^n+2}} = \left(\frac{(n+1)^2+2(n+1)+1}{n^2+2n+1}\right)\left(\frac{3^n+2}{3^{n+1}+2}\right),$$

so pulling out the highest order terms, we have

$$\frac{a_{n+1}}{a_n} = \left(\frac{n^2}{n^2} \cdot \frac{\left(1+\frac{1}{n}\right)^2 + 2\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{1}{n^2}}{1+2\frac{1}{n}+\frac{1}{n^2}}\right)\left(\frac{3^n}{3^{n+1}} \cdot \frac{1+\frac{2}{3^n}}{1+\frac{2}{3^{n+1}}}\right) \to \frac{1}{3}$$

as $n \to \infty$. Because this limit is less than 1, the series converges by the Ratio Test. ●

EXERCISES FOR SECTION 2.6

Exercises 1–4 give various values of

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

In each case, state what you conclude from the Ratio Test about the series $\sum a_n$.

1. $L = 2$

2. $L = 1$

3. $L = 1/2$

4. $L = \infty$

In Exercises 5–16, first compute

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n},$$

and then use the Ratio Test to determine if the given series converge or diverge.

5. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

6. $\sum_{n=1}^{\infty} \frac{7^{n+2}}{2n6^n}$

7. $\sum_{n=1}^{\infty} \frac{n3^n}{n+2}$

$\boxed{\textbf{8.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{n+4^n}$

$\boxed{\textbf{9.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

$\boxed{\textbf{10.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\sqrt{n}}{n!}$

$\boxed{\textbf{11.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$\boxed{\textbf{12.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$\boxed{\textbf{13.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{\sqrt{n!}}$

$\boxed{\textbf{14.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{99^n\sqrt{n!}}$

$\boxed{\textbf{15.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$\boxed{\textbf{16.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

17. Find a sequence $\{a_n\}$ of positive (in particular, nonzero) numbers such that both $\sum a_n$ and $\sum 1/a_n$ diverge.

18. Is there a sequence $\{a_n\}$ satisfying the conditions of the previous problem such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and is not equal to 1?

A stronger test than the Ratio Test, proved by Augustin Louis Cauchy (1789–1857), is the following.

The Root Test. Suppose that $a_n \geqslant 0$ for all n and let $L = \lim_{n\to\infty} \sqrt[n]{a_n}$. The series $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$ and diverges if $L > 1$. (If $L = 1$ then the Root Test is inconclusive.)

Our first task is to prove this result.

◆ **19.** Copying the beginning of the proof of the Ratio Test, give a proof of the Root Test.

Use the Root Test to determine if the series in Exercises 20–26 converge or diverge.

20. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{5n}\right)^{4n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+n}\right)^{n}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n}$

24. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n}$

25. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

26. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

Exercises 27 and 28 show that the Root Test is a stronger test than the Ratio Test.

◆ **27.** Show that the Root Test can handle any series that the Ratio Test can handle by proving that if $L = \lim_{n\to\infty} a_{n+1}/a_n$ exists then $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

◆ **28.** Show that there are series that the Root Test can handle but that the Ratio Test cannot handle by considering the series $\sum a_n$ where

$$a_n = \begin{cases} n/2^n & \text{if } n \text{ is odd,} \\ 1/2^n & \text{if } n \text{ is even.} \end{cases}$$

In some cases where the ratio and root tests are inconclusive, the following test due to Joseph Raabe (1801–1859) can prove useful.

Raabe's Test. Suppose that $\{a_n\}$ is a positive series. If there is some choice of $p > 1$ such that

$$\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$$

for all large n, then $\sum_{n=1}^{\infty} a_n$ converges.

Exercises 29–31 ask you to prove Raabe's Test, while Exercises 32 and 33 consider an application of the test.

◆ **29.** Show that if $p > 1$ and $0 < x < 1$ then

$$1 - px \leqslant (1-x)^p.$$

This is called *Bernoulli's inequality*, after Johann Bernoulli (1667–1748). *Hint:* Set $f(x) = px + (1-x)^p$. Show that $f(0) = 1$ and $f'(x) \geqslant 0$ for $0 < x < 1$. Conclude from this that $f(x) \geqslant 1$ for all $0 < x < 1$.

◆ **30.** Assuming that the hypotheses of Raabe's Test hold and using Exercise 29, show that

$$\frac{a_{n+1}}{a_n} \leqslant \left(1 - \frac{1}{n}\right)^p = \frac{b_{n+1}}{b_n}$$

where $b_n = 1/(n-1)^p$.

◆ **31.** Rewrite the inequality derived in Exercise 30 as

$$\frac{a_{n+1}}{b_{n+1}} \leqslant \frac{b_{n+1}}{b_n},$$

use this to show that $a_n \leqslant M b_n$ for some positive number M and all large n, and use this to prove Raabe's Test.

◆ **32.** Show that the Ratio Test is inconclusive for the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k+2)}.$$

◆ **33.** Use Raabe's Test to prove that the series in Exercise 32 converges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.6

1. The series diverges
3. The series converges
5. $L = 2/3$, so the series converges by the Ratio Test.
7. $L = 3$, so the series diverges by the Ratio Test.
9. $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{n+1} \to 0$ as $n \to \infty$, so the series converges by the Ratio Test.
11. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)(n+1)}{(2n+2)(2n+1)} \to \dfrac{1}{4}$ as $n \to \infty$, so the series converges by the Ratio Test.
13. $\dfrac{a_{n+1}}{a_n} = \sqrt{n+1} \to \infty$ as $n \to \infty$, so the series diverges by the Ratio Test.
15. The ratio here is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} n!}{n^n (n+1)!} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{n^n}{(n+1)^n} = \left(1 + \frac{1}{n}\right)^n .$$

Recall from Example 11 of Section 2.1 that the limit of this ratio is $L = e$, so the series diverges by the Ratio Test because $e > 1$.