1. Taylor Polynomials

1.1. Approximating by Matching Derivatives

Our starting point for exploring series, Taylor polynomials, are at the center of how calculators and computers compute (and graph) most functions. So far you have most likely taken for granted that quantities like $\sin 1/2$ or $e^{0.2}$ can be computed, but have you ever stopped to wonder how?

Consider $\sin 1/2$ first. (Here and in all that follows, angles will be specified in radians, not degrees, although in this case the point remains the same.) If you were asked to compute $\sin 1/2$ to a high degree of accuracy, what would you do? You could of course reach for your protractor, draw a right triangle with angle $1/2$ radians, then measure (with a ruler) the opposite side and hypotenuse and divide them (indeed, this is essentially what the ancient Egyptians and Greeks did), but how sure of your calculation would you be? Would your calculation be accurate to 2 decimal places? What about 5 or 10? More troubling, how could you ever be sure, if you didn't have a calculator to check it against?

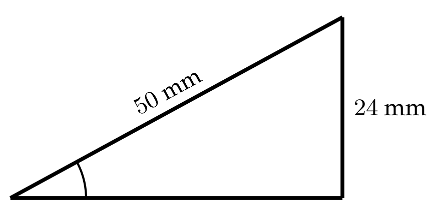


Figure 1.1: Approximating $\sin 1/2$ by drawing a triangle and measuring to the nearest millimeter. This triangle shows that $\sin 1/2 \approx 24/50 = 0.48$. In fact, $\sin 1/2 = 0.4794255386\ldots$

The approach we take here is to find polynomials which approximate $\sin x$.

Example 1. Approximate $\sin x$ near $x = 0$ using a polynomial of degree 5,

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5.$$

Solution. We have to decide on the "best" constants $c_0, c_1, c_2, c_3, c_4, c_5$ to use for this approximation. Let's start with c_0. We have that $p(0) = c_0$ (all the other terms involve an x and so they vanish once we set $x = 0$). Moreover, $\sin 0 = 0$, so it makes sense to set $c_0 = 0$. (If we didn't set $c_0 = 0$, then $p(x)$ and $\sin x$ would disagree at $x = 0$, which is a bad way to start.)

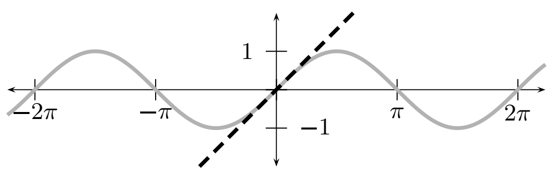
Now what should we do for c_1? This is a much more open-ended question. The approach we will use is to "match derivatives," beginning with the first derivative. Since

$$\begin{aligned} p'(x) &= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 \text{ and} \\ \sin' x &= \cos x, \end{aligned}$$

we see that

$$\begin{aligned} p'(0) &= c_1 \text{ and} \\ \sin' 0 &= 1, \end{aligned}$$

so we set $c_1 = \cos 0 = 1$. Believe it or not, we already have a decent approximation, which is used quite frequently, namely $\sin x \approx x$ for small x.



This approximation gives $\sin 1/2 \approx 0.5$, which is decent, but not as close as our triangle in Figure 1.1.

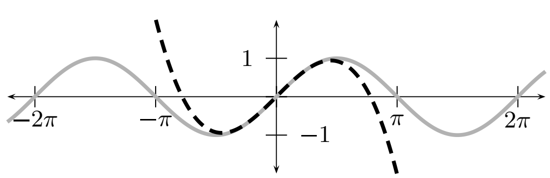
Now that we've found a good value for c_1, we match second derivatives to solve for c_2:

$$\begin{aligned} p''(x) &= 2 \cdot 1 \cdot c_2 + 3 \cdot 2 \cdot c_3x + 4 \cdot 3 \cdot c_4x^2 + 5 \cdot 4 \cdot c_5x^3 \text{ and} \\ \sin'' x &= -\sin x, \end{aligned}$$

we set $p''(0) = 2 \cdot 1 \cdot c_2 = -\sin 0 = 0$, so $c_2 = 0$. For c_3, we match third derivatives:

$$\begin{aligned} p'''(x) &= 3 \cdot 2 \cdot 1 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4x + 5 \cdot 4 \cdot 3 \cdot c_5x^2 \text{ and} \\ \sin''' x &= -\cos x, \end{aligned}$$

This shows that we should have $3 \cdot 2 \cdot 1 \cdot c_3 = -1$, so $c_3 = -1/6$ in order to match third derivatives. Indeed, by ignoring the c_4 and c_5 terms we get an even better approximation to $\sin x$ near $x = 0$, $x - x^3/6$, plotted below.



This approximation gives $\sin 1/2 \approx 0.47916666\ldots$. This is already closer to the true value of $\sin 1/2$ than our triangle before.

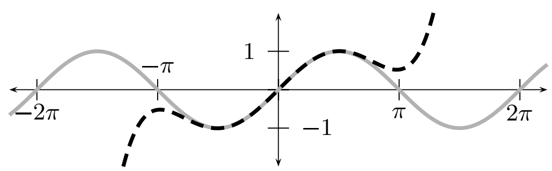
Using the same approach for c_4, we see

$$\begin{aligned} p^{(4)}(x) &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot c_4 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot c_5 x \text{ and} \\ \sin^{(4)} x &= \sin x, \end{aligned}$$

so $p^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot c_4 = \sin 0 = 0$. To conclude our example, we match fifth derivatives:

$$\begin{aligned} p^{(5)}(x) &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot c_5 \text{ and} \\ \sin^{(5)} x &= \cos x, \end{aligned}$$

so $c_5 = \dfrac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{1}{120}$. Comparing the graphs of $\sin x$ and $x - x^3/6 + x^5/120$, we see that this is an ever better approximation:



This approximation gives $\sin 1/2 \approx 0.4794270834$, which is correct to 5 decimal places. ●

Example 1 is our first encounter with *Taylor polynomials*. In fact, the polynomial $p(x)$ that we computed is known as the *Taylor polynomial of degree* 5 *for the function* $\sin x$ *centered at* 0. In general, the Taylor polynomial of degree n for the function $f(x)$ centered at a is the polynomial $T_n(x)$ that matches $f(x)$ and its first n derivatives at $x = a$:

> **Taylor Polynomials.** Suppose that $f(x)$ has n derivatives at the point $x = a$. Then the Taylor polynomial of degree n for $f(x)$ centered at a, denoted $T_n(x)$, is the unique polynomial of degree n which satisfies
>
> $$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\vdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

Note that we write $T_n(x)$ for the Taylor polynomial of degree n no matter where it is centered, i.e., no matter what a happens to be. These polynomials are named after the English mathematician Brook Taylor (1685–1731), who discussed them in a 1715 work. However, the importance of Taylor polynomials was not realized until after Taylor's death, when the Italian mathematician and astronomer Joseph Louis Lagrange (1736–1813) declared them to be "the main foundation of differential calculus."

We have just defined Taylor polynomials in terms of their most important property — they match derivatives. It is possible to state this definition in a more formulaic manner. We first state the formula, then explain all the terms involved, then prove it.

Formula for Taylor Polynomials. Suppose that $f(x)$ has n derivatives at the point $x = a$. Then the Taylor polynomial of degree n for $f(x)$ centered at a is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The exclamation marks in this theorem may be new to you. These are called *factorials*. The factorial of the integer n is $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$; it is the product of all of the integers between 1 and n (inclusive). The factorial function grows *very* quickly, much more quickly than any polynomial or even exponential function. This will be important.

Proof. Proving this formula is quite easy. Let

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Then we have

$$\begin{aligned}
T_n(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
T_n'(x) &= f'(a) + \frac{f''(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} \\
T_n''(x) &= f''(a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2} \\
&\vdots \\
T_n^{(k)} &= f^{(k)}(a) + \cdots \\
&\vdots \\
T_n^{(n)} &= f^{(n)}(a).
\end{aligned}$$

Substituting $x = a$ into these equalities verifies that the formula given satisfies the definition of the Taylor polynomial of degree n for $f(x)$ centered at a. $\square$

We define $0! = 1$ in order to make this formula easier to write. We also define $f^{(0)}(a)$, the "zeroth derivative of f at a", to be just $f(a)$. This lets us express the Taylor polynomial as

Formula for Taylor Polynomials, Summation Form.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Here we have used yet another new piece of notation a capital Greek sigma $\sum$. This symbol stands for "sum," and in the above it means "add together $\dfrac{f^{(k)}(a)}{k!}(x-a)^k$ for each integer k from 0 to n."

Example 2. Compute the Taylor polynomial of degree 4 for the function $f(x) = e^x$ centered at $x = 0$.

Solution. We begin by constructing a table of derivatives:

$$\begin{array}{lcl|lcl}
\hline
f(x) & = & e^x & f(0) & = & 1 \\
f'(x) & = & e^x & f'(0) & = & 1 \\
f''(x) & = & e^x & f''(0) & = & 1 \\
& \vdots & & \vdots & &
\end{array}$$

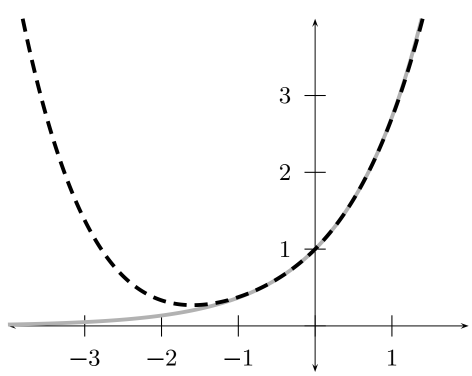
This table demonstrates that all the derivatives of e^x at 0 are equal to 1. So the Taylor polynomial of degree 4 for e^x centered at $a = 0$ is:

$$T_4(x) = \frac{1}{0!}(x-0)^0 + \frac{1}{1!}(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4.$$

Of course we would never want to write it that way, instead simplifying to

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Plotting this against $f(x) = e^x$,



we see that it is a good approximation for e^x near $x = 0$. ●

Example 3. Compute the Taylor polynomial of degree 4 for the function $f(x) = \sin x$ centered at $a = \pi/4$.

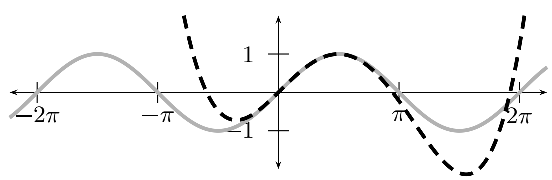
Solution. Again we begin with a table of derivatives:

$f(x)$	$=$	$\sin x$	$f(\pi/4)$	$=$	$\sqrt{2}/2$
$f'(x)$	$=$	$\cos x$	$f'(\pi/4)$	$=$	$\sqrt{2}/2$
$f''(x)$	$=$	$-\sin x$	$f''(\pi/4)$	$=$	$-\sqrt{2}/2$
$f'''(x)$	$=$	$-\cos x$	$f'''(\pi/4)$	$=$	$-\sqrt{2}/2$
$f^{(4)}(x)$	$=$	$\sin x$	$f^{(4)}(\pi/4)$	$=$	$\sqrt{2}/2$

Therefore the Taylor polynomial is

$$T_4(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!}\left(x - \frac{\pi}{4}\right)^4 .$$

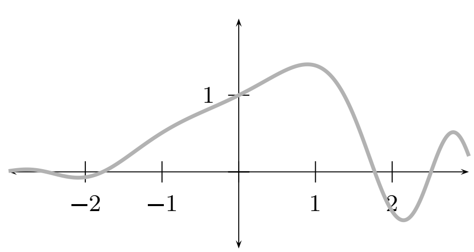
Comparing this to the plot of $\sin x$ shows that it is quite a good approximation near $\pi/4$:



In fact, $T_3(0.5) = 0.4792417350\ldots$, which is a closer approximation to $\sin 1/2$ than the Taylor polynomial of degree 3 centered at $a = 0$. ●

EXERCISES FOR SECTION 1.1

Explain why none of the polynomials in Exercises 1–4 are the Taylor polynomials of the function shown below.



1. $\frac{1}{2} + \frac{x}{2} + \frac{x^2}{8}$ (centered at 0)

2. $1.4 - 0.3(x-1) + 1.6(x-1)^2$ (centered at 1)

3. $0.5 - 0.6(x+1) - 0.2(x+1)^2$ (centered at -1)

4. $-0.5 + 1.5(x-2) + 4.4(x-2)^2$ (centered at 2)

Exercises 5–8 give various Taylor polynomials for functions $f(x)$ centered at 2. For each function, compute $f'''(2)$.

5. $(x-2)+\dfrac{(x-2)^3}{6}+\dfrac{(x-2)^5}{24}$

6. $2(x-2)-\dfrac{3(x-2)^2}{2}+2(x-2)^3-\dfrac{31(x-2)^4}{12}$

7. $1+\dfrac{(x-2)^2}{2}+\dfrac{5(x-2)^4}{24}$

8. $(x-2)-(x-2)^2+\dfrac{5(x-2)^3}{6}-\dfrac{5(x-2)^4}{6}$

Compute the Taylor polynomials of degree 4 centered at 0 for the functions in Exercises 9–18.

9. $f(x)=\dfrac{1}{1-x}$

10. $f(x)=\cos 2x$

11. $f(x)=x\sin x$

12. $f(x)=e^{x^2}$

13. $f(x)=\sqrt[3]{1+x}$

14. $f(x)=2-x+3x^3$

15. $f(x)=2-x+8x^9$

16. $f(x)=(1+x)^6$

17. $f(x)=x\ln(1+x)$

18. $f(x)=\cos^2 x$

Compute the Taylor polynomials of degree 3 centered at π for the functions in Exercises 19–22.

19. $f(x)=\dfrac{\sin x}{x}$

20. $f(x)=\ln x$

21. $f(x)=\frac{1}{x}$

22. $f(x)=x+3x^2$

23. Explain why the Taylor polynomial of degree 1 for the function $f(x)$ centered at a is the equation of the tangent line to the graph of f at a.

24. Explain how you know that $\sin x$ and $\cos x$ are not polynomials.

25. Explain how you know that e^x is not a polynomial.

ANSWERS TO SELECTED EXERCISES, SECTION 1.1

1. The Taylor polynomial does not match the function at the center
3. The function is increasing at the center, but the first derivative of the Taylor polynomial is negative here
5. $f'''(2) = 1$
7. $f'''(2) = 0$
9. $T_4(x) = 1 + x + x^2 + x^3 + x^4$
11. $T_4(x) = x^2 - \dfrac{x^4}{6}$
13. $T_4(x) = 1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{5x^3}{81} - \dfrac{10x^4}{243}$
15. $T_4(x) = 2 - x$
17. $T_4(x) = x^2 - \dfrac{x^3}{2} + \dfrac{x^4}{3}$
19. $T_3(x) = -\frac{1}{\pi}(x-\pi) + \frac{1}{\pi^2}(x-\pi)^2 + \frac{\pi^2-6}{6\pi^3}(x-\pi)^3$
21. $T_3(x) = \frac{1}{\pi} - \frac{1}{\pi^2}(x-\pi) + \frac{1}{\pi^3}(x-\pi)^2 - \frac{1}{\pi^4}(x-\pi)^3$
23. Because the Taylor polynomial of degree 1, $T_1(x) = f(a) + f'(a)x$, matches the function and its first derivative at $x = a$.