NAME : ______________________________

SECTION : (circle one) Weber (10 Hour) Vatter (11 hour) Vatter (12 hour)

Math 8

6 December 2008

Final Exam

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have three hours and you should attempt all problems.

- **Except for the multiple choice questions, you must show all work and give a reason (or reasons) for your answer. A CORRECT ANSWER WITH INCORRECT WORK WILL BE CONSIDERED WRONG.**
- ***Print*** your name in the space provided and circle your instructor's name, **both on this exam and on the answer sheet.**
- Calculators or other computing devices are not allowed.
- Use the blank page at the end of the exam for scratch work.

1. (10) The functions $\sinh x$ and $\cosh x$ satisfy the following equalities:

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x.$$

Use this information to evaluate

$$\int \sinh^3 x \cosh^3 x \, dx.$$

2. (12) Evaluate $\int e^x \sin x \, dx$.

3. (12) Find the Maclaurin series (Taylor series centered at $x = 0$) for the function

$$f(x) = 2x^3 \ln(1 + 2x^2).$$

Then, find the radius (not the interval) of convergence of this series. (Suggestion: find the Maclaurin series for $\ln(1 + x)$ first.)

4. (10) For the following five series, choose their sums. The answers are **A**. $12{}^1\!/\!_2$

B. $\sqrt{2}$

C. e^2

D. $\sqrt{5}$

E. ${}^3\!/\!_2$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}.$$

$$\sum_{n=0}^{\infty} \binom{1/2}{n} 2^{2n}.$$

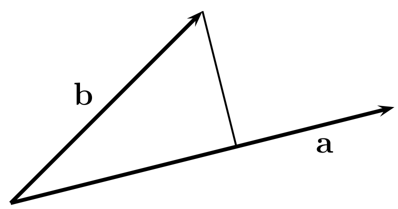
$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2^{4n-1}(2n)!}.$$

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}+3^n}{5^n}.$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

5. (12) Find the line of intersection of the planes $x + y + z = 12$ and $2x + 3y + z = 2$.

6. (12) The vectors in the figure below are given by $\mathbf{a} = \langle 4, 1 \rangle$ and $\mathbf{b} = \langle 2, 2 \rangle$. Find the area of the right triangle.



Note: you should not leave unevaluated trigonometric functions in your answer.

7. (10) Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

8. (12) Compute the tangent plane to the function $f(x, y) = x \cos y + x^2 \sin y$ at the point $(1, \pi/4)$.

9. (15) Find the local maximum and minimum values and saddle point(s) of the function

$$f(x, y) = x^3 - 12xy + 8y^3.$$

10. (15) This question has three short answer parts.

(a) What is the domain of the function $f(x, y) = \sqrt{y - x}$?

(b) Suppose $\nabla f(a, b) = \sqrt{3}\mathbf{i} - \mathbf{j}$. Find a direction (unit vector) $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}} f(a, b) = 0$.

(c) Can you conclude anything about $f(a, b)$ if f and its first and second derivatives are continuous, (a, b) is a critical point, and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign?

11. (4) In order to evaluate the integral $\int \frac{x^2-3}{\sqrt{4+9x^2}}\,dx$, which substitution would you make?

A. $x = 2\sec\theta$

B. $x = \frac{4}{9}\sec\theta$

C. $x = \frac{2}{3}\tan\theta$

D. $x = \frac{4}{9}\tan\theta$

E. $x = \frac{2}{3}\sec\theta$

12. (4) After making the substitution $x = 2\sin\theta$ in order to evaluate $\int \frac{x}{\sqrt{4-x^2}}\,dx$, the integral becomes $\int 2\sin\theta\,d\theta = -2\cos\theta + C$. Express this in terms of x.

A. $-\sqrt{4-x^2}+C$

B. $-\sqrt{4+x^2}+C$

C. $-2\frac{x}{\sqrt{4-x^2}}+C$

D. $-2\frac{x}{\sqrt{4+x^2}}+C$

E. $-\frac{x}{2}+C$

13. (4) Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and that f'' is continuous. What is $\displaystyle\int_1^4 x f''(x)\, dx$?

A. 0

B. 1

C. 2

D. 5

E. 12

14. (4) What is $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n}$?

A. 0

B. $^1/_e$

C. 1

D. e

E. ∞

15. (4) Suppose $\sum_{n=1}^{\infty} a_n = 3$ and s_n is the nth partial sum of the series. Which of the following statements is true?

A. $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n$ need not exist

B. $\lim_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} s_n = 0$

C. $\lim_{n\to\infty} a_n = 3$ and $\lim_{n\to\infty} s_n = \infty$

D. $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$

E. $\lim_{n\to\infty} a_n$ need not exist and $\lim_{n\to\infty} s_n = 3$

16. (4) Suppose that $0 \leq a_n \leq 1$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Which of the following related series necessarily converge?

I	II	III	IV
$\sum_{n=1}^{\infty} (a_n)^2$	$\sum_{n=1}^{\infty} e^{a_n}$	$\sum_{n=1}^{\infty} \sqrt{a_n}$	$\sum_{n=1}^{\infty} \sin(n) a_n$

A. I and III only

B. I and IV only

C. I, II, and III only

D. III and IV only

E. I, III, IV only

17. (4) Which of the following series converge?

I	II	III	IV
$\sum_{n=1}^{\infty}(-1)^n \frac{2n-4}{\sqrt{4n^3+n}}$	$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$	$\sum_{n=1}^{\infty}(1+{}^1\!/\!_n)^{-n}$	$\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$

A. I and IV only

B. I, II, and III only

C. I, II, and IV only

D. II and IV only

E. all of the series converge

18. (4) What is the coefficient of x^{12} in $(1+3x^2)^{27}$?

A. $\binom{27}{12}$

B. $\binom{27}{12}3^{12}$

C. $\binom{27}{6}3^{12}$

D. $\binom{27}{12}3^{6}$

E. $\binom{27}{6}3^{6}$

19. (4) Let $\mathbf{a} = \langle 3, -1, 2\rangle$ and $\mathbf{b} = \langle -2, 7, 3\rangle$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is...

A. acute

B. obtuse

C. right

20. (4) What is the area of the parallelogram with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 4, 3)$, and $(1, 4, 3)$?

A. $-3\mathbf{j} + 4\mathbf{k}$

B. $3\mathbf{j} - 4\mathbf{k}$

C. 5

D. 7

E. 9

21. (4) Suppose the scalar projection of $\mathbf{v}$ onto $\mathbf{w}$ is $-2\sqrt{5}$. If $w = \langle 1, 2 \rangle$, then what is $\text{proj}_{\mathbf{w}}\mathbf{v}$?

A. $-2\sqrt{5}$

B. $2\sqrt{5}$

C. $\langle -2, -4 \rangle$

D. $\langle -2\sqrt{5}, -4\sqrt{5} \rangle$

E. $\langle 2, 4 \rangle$

22. (4) Compute the position vector for a particle which passes through the origin at time $t = 0$ and has velocity vector $\mathbf{v}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t^2\mathbf{k}$.

A. $-\cos t\mathbf{i} + \sin t\mathbf{j} + (t^3/3)\mathbf{k}$

B. $\sin t\mathbf{i} + (\cos t - 1)\mathbf{j} + 2t\mathbf{k}$

C. $(\cos t - 1)\mathbf{i} - \sin t\mathbf{j} + 2t\mathbf{k}$

D. $(1 - \cos t)\mathbf{i} + \sin t\mathbf{j} + (t^3/3)\mathbf{k}$

E. $-\cos t\mathbf{i} + \sin t\mathbf{j} + (t^3/3)\mathbf{k} + 1$

23. (4) Let $\mathbf{r}(t)$ denote the position of a particle, $\mathbf{v}(t)$ denote its velocity function, and $\mathbf{a}(t)$ denote its acceleration function. Which of the following integrals gives its arc length from $t = 0$ to $t = 1$?

A. $\displaystyle\int_0^1 \mathbf{r}(t) \bullet \mathbf{r}(t)\, dt$

B. $\displaystyle\int_0^1 \sqrt{\mathbf{r}(t) \bullet \mathbf{r}(t)}\, dt$

C. $\displaystyle\int_0^1 \mathbf{v}(t) \bullet \mathbf{a}(t)\, dt$

D. $\displaystyle\int_0^1 \sqrt{\mathbf{v}(t) \bullet \mathbf{v}(t)}\, dt$

E. $\displaystyle\int_0^1 \mathbf{r}(t) \bullet \mathbf{v}(t)\, dt$

24. (4) Suppose that $f(x, y)$ and all of its partial derivatives are continuous. How many distinct third order partial derivatives can f have?

A. 1

B. 2

C. 3

D. 4

E. 8

25. (4) Suppose that $f(s,t) = g(u(s,t), v(s,t))$ where:

$$\begin{array}{rclrclrcl} u(1,0) & = & 7 & u_s(1,0) & = & 2 & u_t(1,0) & = & 3 \\ v(1,0) & = & 2 & v_s(1,0) & = & 4 & v_t(1,0) & = & 2 \\ g_u(7,2) & = & 3 & g_v(7,2) & = & -2 & & & \end{array}$$

What is $f_s(1,0)$?

A. -5

B. -2

C. 0

D. 2

E. 5

26. (4) Suppose that $f_x(4,3) = 30$ and $f_y(4,3) = -40$. Compute the derivative of f at $(4,3)$ in the direction of the vector $\langle 3, 4 \rangle$.

A. -70

B. -14

C. -10

D. 10

E. $\sqrt{1348}$

27. (4) Suppose that $\bigtriangledown f(a,b) = 3\mathbf{i} - 4\mathbf{j}$. What is the greatest directional derivative of f at (a,b)?

A. $-1/5$

B. $1/5$

C. $7/5$

D. $18/5$

E. 5

28. (4) Which of the following statements are true if $f(x,y)$ is differentiable at (a,b)?

I	If $\mathbf{u}$ is a unit vector, the derivative of f at (a,b) in the direction of $\mathbf{u}$ is $(f_x(a,b)\mathbf{i} + f_y(a,b)\mathbf{j}) \bullet \mathbf{u}$.
II	The derivative at (a,b) in the direction $\mathbf{u}$ is a vector.
III	The directional derivative of f at (a,b) has its greatest value in the direction of $\bigtriangledown f(a,b)$.
IV	At (a,b), the vector $\bigtriangledown f(a,b)$ is normal to the curve $f(x,y) = f(a,b)$.

A. I and III

B. II and III

C. I, III, and IV

D. II, III, and IV

E. All of these statements are true.

29. (4) Which of the following is the equation for the tangent plane to the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{10} = 34$$

at the point $(8, 8, -10)$?

A. $2(x-8) + 4(y-8) - 5(z+10) = 0$

B. $\dfrac{x-8}{4} + \dfrac{y-8}{8} + \dfrac{z+10}{10} = 0$

C. $4(x-8) + 8(y-8) - 10(z+10) = 0$

D. $\dfrac{x-8}{2} + \dfrac{y-8}{4} + \dfrac{z+10}{5} = 0$

E. $4(x-8) + 2(y-8) - 2(z+10) = 0$

30. (4) Suppose that the function f has critical points at $(0, 0)$ and $(1, 1)$ (plus possibly other critical points), and that its second derivatives are given by

$$f_{xx} = 4x + 2 \qquad f_{xy} = -3 \qquad f_{yy} = 2y + 3$$

Classify these critical points.

A. $f(0, 0)$ is a local minimum, $f(1, 1)$ is a local minimum

B. $f(0, 0)$ is a saddle point, $f(1, 1)$ is a local maximum

C. $f(0, 0)$ is a saddle point, $f(1, 1)$ is a local minimum

D. $f(0, 0)$ is a local maximum, $f(1, 1)$ is a saddle point

E. $f(0, 0)$ is a saddle point, $f(1, 1)$ is a saddle point