1. (6) For each of the following six Taylor series centered at $x = 0$, write down the corresponding function. You need not show work, and each of the answers is one of the following:

A	$\sin(2x)$
B	$\cos(2x)$
C	$x\cos(2x)$
D	$\sqrt{1/4 + x^2}$
E	$\frac{1}{4-x^2}$
F	$\int_0^{2x} e^{-t^2}\,dt$

(a) $\displaystyle\sum_{n=0}^{\infty}(-4)^n\frac{x^{2n+1}}{2n!}$. C

(b) $\displaystyle\sum_{n=0}^{\infty}(-4)^n\frac{x^{2n}}{2n!}$. B

(c) $\displaystyle\sum_{n=0}^{\infty}2(-4)^n\frac{x^{2n+1}}{(2n+1)n!}$. F

(d) $\displaystyle\sum_{n=0}^{\infty}2(-4)^n\frac{x^{2n+1}}{(2n+1)!}$. A

(e) $\displaystyle\sum_{n=0}^{\infty}\binom{1/2}{n}2^{n-1}x^n$. D

(f) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{2^{n+2}}$. E

2. (10) Compute the first four terms of the Taylor series for $\ln(\sec(x))$ centered at $x = \pi/4$.

- $\tan \frac{\pi}{4} = 1$
- $\sec \frac{\pi}{4} = \sqrt{2}$

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\ln(\sec(x))$	$\ln(\sqrt{2})$
1	$\frac{\sec x \tan x}{\sec x} = \tan x$	1
2	$\sec^2 x$	2
3	$2 \sec^2 x \tan x$	4

$$T_3(x) = \ln(\sqrt{2}) + \left(x - \frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3$$

3. (6) Determine whether the following are true or false in 3-space. (No work is required.)

(a) Two planes either intersect or are parallel.

T

(b) Two lines either intersect or are parallel.

F (skew)

(c) Two lines parallel to a plane are parallel.

F (x-axis, y-axis, xy-plane)

(d) Two lines orthogonal to a third line are parallel.

F (x-, y-, and z-axes)

(e) A plane and a line either intersect or are parallel.

T

(f) Two planes orthogonal to a third plane are parallel.

F (coordinate planes)

4. (8) Find the area of the parallelogram whose vertices are $(-1, 2, 0)$, $(0, 4, 2)$, $(2, 1, -2)$, and $(3, 3, 0)$.

2 vectors with same tail:

$\langle 3+1, 3-2, 0-0\rangle = \langle 4, 1, 0\rangle$

$\langle 0+1, 4-2, 2-0\rangle = \langle 1, 2, 2\rangle$

cross-product:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 0 \\ 1 & 2 & 2 \end{vmatrix} = (2-0)\vec{i} + (0-8)\vec{j} + (8-1)\vec{k}$$

$$= \langle 2, -8, 7\rangle$$

length of cross-product:

$\sqrt{4+64+49} = \sqrt{117} =$ area

5. (8) For each of the following pairs of lines determine whether they are parallel, intersecting, or skew.

(a) $\begin{cases} x = 3t - 2, & y = t + 3, & z = 5t - 3 \\ x = -6s - 5, & y = -2s, & z = -10s - 6 \end{cases}$

direction vectors $\langle 3, 1, 5 \rangle$

$\langle -6, -2, -10 \rangle = -2\langle 3, 1, 5 \rangle$

parallel

(b) $\begin{cases} x = 3t - 2, & y = t + 3, & z = 5t - 3 \\ x = s - 4, & y = 2s, & z = 4s - 6 \end{cases}$

$\langle 3, 1, 5 \rangle$
$\langle 1, 2, 4 \rangle$ not parallel

find s, t such that x, y are equal

$3t - 2 = s - 4$

$t + 3 = 2s$

$3(2s - 3) - 2 = s - 4$

$6s - 11 = s - 4$

$5s = 7$

$s = 7/5$

$t = \frac{14}{5} - 3 = -\frac{1}{5}$

check if z also matches:

$5(-\frac{1}{5}) - 3 = -1 - 3 = -4$

$4(\frac{7}{5}) - 6 = \frac{28}{5} - \frac{30}{5} = \frac{-2}{5}$

no

not parallel, not intersecting

ie. skew

6. (9) Find the equation of the plane which passes through the point $(2, -3, 1)$ and contains the line

$$x = 3t - 2, \quad y = t + 3, \quad z = 5t - 3 \ .$$

$\vec{r}_0 = \langle 2, -3, 1 \rangle$

for $\vec{n}$, cross 2 vectors in the plane:

$\langle 3, 1, 5 \rangle$ (direction vector of line)

$\langle 2+2, -3-3, 1+3 \rangle = \langle 4, -6, 4 \rangle$

(given point − point on line)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 5 \\ 4 & -6 & 4 \end{vmatrix} = (4+30)\vec{i} + (20-12)\vec{j} + (-18-4)\vec{k}$$

$$= \langle 34, 8, -22 \rangle$$

$\vec{n} \cdot \vec{r}_0 = 68 - 24 - 22 = 22$

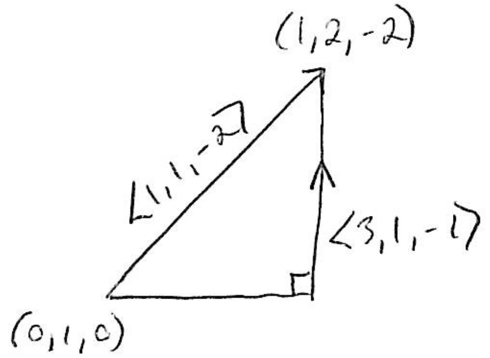
$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$ is

$34x + 8y - 22z = 22$ or $17x + 4y - 11z = 11$

7. (9) Compute the distance from the point $(1, 2, -2)$ to the plane given by $3x + y - z = 1$.

point on plane: $(0, 1, 0)$

normal vector: $\langle 3, 1, -1 \rangle$



project $\langle 1, 1, -2 \rangle = \vec{v}$

onto $\langle 3, 1, -1 \rangle = \vec{n}$

want scalar projection.

$$\text{comp}_{\vec{n}} \vec{v} = \frac{\vec{v} \cdot \vec{n}}{|\vec{n}|}$$

$$= \frac{3+1+2}{\sqrt{9+1+1}} = \frac{6}{\sqrt{11}} = \text{Distance}$$

8. (8) Compute the position vector for a particle which passes through the origin at time $t = 0$ and has velocity vector

$$\mathbf{V}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}.$$

$$\int \vec{v}(t)\,dt = \langle t^2, -\cos t, \sin t \rangle + \vec{C}$$

$$\langle 0^2, -\cos 0, \sin 0 \rangle + \vec{C} = \langle 0, 0, 0 \rangle$$

$$\langle 0, -1, 0 \rangle + \vec{C} = \langle 0, 0, 0 \rangle$$

$$\vec{C} = \langle 0, 1, 0 \rangle$$

position vector is $\langle t^2, 1 - \cos t, \sin t \rangle$

9. (8) Show that if a particle moves at constant speed, then its velocity and acceleration vectors are orthogonal. (Hint: consider the derivative of $\mathbf{v} \bullet \mathbf{v}$.)

by definition,

$$\frac{d}{dt}\vec{v}\cdot\vec{v} = \vec{v}\cdot\vec{v}' + \vec{v}'\cdot\vec{v}$$

$$= 2\vec{v}\cdot\vec{v}' = 2\vec{v}\cdot\vec{a}$$

also

$$\frac{d}{dt}\vec{v}\cdot\vec{v} = \frac{d}{dt}|\vec{v}|^2$$

$= 0$ because $|\vec{v}|$ and hence $|\vec{v}|^2$ are constant.

$2\vec{v}\cdot\vec{a} = 0$ means $\vec{v}\cdot\vec{a} = 0$ and so $\vec{v} \perp \vec{a}$.

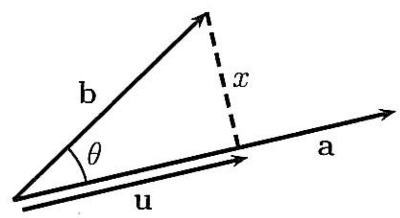
Note $\vec{v} = \langle \cos t, \sin t \rangle$ has constant speed 1

and acceleration $\langle -\sin t, \cos t \rangle$, so

$\vec{v}\cdot\vec{a} = -\cos t \sin t + \sin t \cos t = 0$

though neither $\vec{v}$ nor $\vec{a}$ is $\vec{0}$.

10. (8) Consider the vectors $\mathbf{a} = \langle 4, 1\rangle$ and $\mathbf{b} = \langle 2, 2\rangle$, shown below. Compute $\cos\theta$, $\mathbf{u}$, and the length x.



Quantities we'll want:

$|\vec{a}| = \sqrt{16+1} = \sqrt{17}$

$|\vec{b}| = \sqrt{4+4} = 2\sqrt{2}$

$\vec{a}\cdot\vec{b} = 8+2 = 10$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{10}{2\sqrt{2}\sqrt{17}} = \frac{5}{\sqrt{34}}$$

$$\vec{u} = \text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{10}{17}\langle 4,1\rangle = \left\langle \frac{40}{17}, \frac{10}{17}\right\rangle$$

$$\left(\text{or: } \vec{u} = |\vec{u}|\frac{\vec{a}}{|\vec{a}|} = |\vec{b}|\cos\theta\frac{\vec{a}}{|\vec{a}|} = 2\sqrt{2}\cdot\frac{10}{2\sqrt{2}\sqrt{17}}\cdot\frac{1}{\sqrt{17}}\langle 4,1\rangle\right)$$

$$x = \sqrt{|\vec{b}|^2 - |\vec{u}|^2}$$

$$= \sqrt{8 - \frac{100}{17}} = \sqrt{36/17} = \frac{6}{\sqrt{17}}$$

11. (8) Consider the curve defined by

$$\mathbf{r}(t) = \langle 4\sin ct, 3ct, 4\cos ct \rangle .$$

What value of c makes the arc length of the space curve traced by $\mathbf{r}(t)$, $0 \le t \le 1$, equal to 10?

$$\text{length} = \int_0^1 |\vec{r}\,'(t)|\, dt$$

$$\vec{r}\,'(t) = \langle 4c\cos ct,\ 3c,\ -4c\sin ct \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{16c^2\cos^2(ct) + 9c^2 + 16c^2\sin^2(ct)}$$

$$= \sqrt{16c^2 + 9c^2} = \sqrt{25c^2} = 5c$$

$$\int_0^1 5c\, dt = 5ct\Big|_0^1 = 5c$$

$$5c = 10 \Rightarrow \underline{c = 2}$$

12. (12) This question has 4 short answer parts.

(a) The function f is defined by $f(x) = \sum_{n=0}^{\infty} (-3)^n \frac{(x-2)^n}{(n^2+1)n!}$. Compute $f^{(38)}(2)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$f^{(n)}(2) = \frac{(-3)^n}{n^2+1} \quad \text{so} \quad f^{(38)}(2) = \frac{3^{38}}{38^2+1}$$

(b) What is the coefficient of x^{12} in the expansion of $(1+2x)^{24}$?

$$(1+2x)^{24} = \sum_{n=0}^{\infty} \binom{24}{n}(2x)^n$$

at $n=12$, coeff. is $\binom{24}{12}2^{12}$

(c) Is the angle between the vectors $\mathbf{a} = \langle 3, -1, 2\rangle$ and $\mathbf{b} = \langle 2, 2, 4\rangle$ acute, obtuse, or right?

$\vec{a} \cdot \vec{b} = 6 - 2 + 8 = 12$

$12 > 0$ so <u>acute</u>

(d) If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero vectors and $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a} \times \mathbf{b}|$, what can you say about the relationship between $\mathbf{a}$ and $\mathbf{b}$?

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

if $\vec{a} \cdot \vec{b} = |\vec{a} \times \vec{b}|$ we must have $\cos\theta = \sin\theta$

ie the angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$.