1. (10) Evaluate

$$\int x\cos^2(2x)\,dx.$$

$u = x$ $\quad dv = \cos^2(2x)\,dx = \left[\frac{1}{2} + \frac{1}{2}\cos(4x)\right)dx$

$du = dx$ $\quad v = \frac{1}{2}x + \frac{1}{8}\sin(4x)$

$uv \int v\,du =$

$\frac{1}{2}x^2 + \frac{1}{8}x\sin(4x) - \int\left[\frac{1}{2}x + \frac{1}{8}\sin(4x)\right]dx$

$= \frac{1}{2}x^2 + \frac{1}{8}x\sin(4x) - \frac{1}{4}x^2 + \frac{1}{32}\cos(4x) + C$

$= \frac{1}{4}x^2 + \frac{1}{8}x\sin(4x) + \frac{1}{32}\cos(4x) + C$

2. (6) Evaluate

$$\int \sec^4(x)\tan^4(x)\,dx$$

$$= \int \sec^2 x \tan^4 x \underbrace{\sec^2 x \, dx}_{du \text{ for } u = \tan x}$$

$$= \int (1+\tan^2 x)\tan^4 x \sec^2 x \, dx$$

$$= \int (u^4 + u^6)\,du = \frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

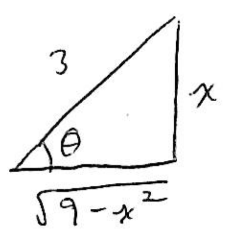
3. (8) Evaluate

$$\int \frac{dx}{(9-x^2)^{3/2}}.$$

$x = 3\sin\theta \qquad dx = 3\cos\theta \, d\theta$

$$\int \frac{3\cos\theta \, d\theta}{(9\cos^2\theta)^{3/2}} = \int \frac{3\cos\theta \, d\theta}{27\cos^3\theta} = \frac{1}{9}\int \frac{1}{\cos^2\theta} \, d\theta$$

$$= \frac{1}{9}\int \sec^2\theta \, d\theta = \frac{1}{9}\tan\theta + C$$



$\sin\theta = \frac{x}{3} \qquad \tan\theta = \frac{x}{\sqrt{9-x^2}}$

$$= \frac{x}{9\sqrt{9-x^2}}$$

4. (8) Find the value of the series

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n}.$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$$ (if both series converge)

geometric, $r = \frac{2}{5}$ or $\frac{3}{5}$, both < 1

$$= \frac{4/5}{1-2/5} + \frac{3/5}{1-3/5} = \frac{4}{3} + \frac{3}{2} = \frac{17}{6}$$

5. (8) Prove whether the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n}$$

converges or diverges.

positive, continuous, decreasing function $f(x) = \frac{1}{x \ln x}$

integral test

$$\int_3^{\infty} \frac{1}{x \ln x} dx = \lim_{t \to \infty} \int_{\ln 3}^{t} u^{-1} du = \lim_{t \to \infty} (\ln t - \ln \ln 3)$$

$u = \ln x$

$du = \frac{1}{x} dx$

$$= \infty$$

integral diverges, so series also diverges.

6. (7) Determine if the series

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)\cos^2(n)}{n^3+2n}$$

converges. Mention any test(s) that you might use and verify that they are applicable.

comparison – all terms are positive

$$0 \leq \frac{\sin^2 n \cos^2 n}{n^3+2n} \leq \frac{1}{n^3+2n} < \frac{1}{n^3}$$

$\sum \frac{1}{n^3}$ convergent p-series so the given series also converges.

7. (7) Determine if the series

$$\sum_{n=2}^{\infty} \frac{n^2+2n-1}{\sqrt{n^5-4}}$$

converges. Mention any test(s) that you might use and verify that they are applicable.

compare to $\sum \frac{n^2}{n^{5/2}} = \sum \frac{1}{\sqrt{n}}$ divergent p-series.

all terms are positive since $n \geq 2$.

(A) limit comparison

$$\lim_{n\to\infty} \frac{(n^2+2n-1)}{\sqrt{n^5-4}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n^{5/2}+2n^{3/2}-n^{1/2}}{\sqrt{}}$$

leading terms of equal degree $= \frac{1}{\sqrt{1}} = 1 \qquad 0<1<\infty$

limit comparison gives divergence of our series.

(B) direct comparison

since $n \geq 1$

$$\frac{n^2+2n-1}{\sqrt{n^5-4}} > \frac{n^2+2n-1}{n^{5/2}} > \frac{n^2}{n^{5/2}} = \frac{1}{\sqrt{n}} > 0$$

direct comparison gives divergence of our series

8. (8) Determine if the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

converges. Mention any test(s) that you might use and verify that they are applicable.

ratio test (no conditions to check)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}\right| = \frac{(n+1)^2}{(n+1)\,n^2} = \frac{n^2+2n+1}{n^3+n^2}$$

$$\lim_{n\to\infty} \frac{n^2+2n+1}{n^3+n^2} = 0$$

$0<1$ so ratio test gives absolute convergence.

9. (10) Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{6^n\sqrt{n}}.$$

ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}\,6^n\sqrt{n}}{6^{n+1}\sqrt{n+1}\,(x-2)^n}\right|$$

$$= \frac{|x-2|\sqrt{n}}{6\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{|x-2|\sqrt{n}}{6\sqrt{n+1}} = \frac{|x-2|}{6}$$

need $\frac{|x-2|}{6} < 1$

$-4 < x < 8$

$R = 6$

$\boxed{\text{OR}}$

root test

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{|x-2|^n}{6^n\sqrt{n}}} = \frac{|x-2|}{6\,n^{1/2n}}$$

$$\lim_{n\to\infty} \frac{|x-2|}{6\,n^{1/2n}} = \frac{|x-2|}{6}$$

need $\frac{|x-2|}{6} < 1$

Check endpoints:

$x = -4$

$$\sum_{n=1}^{\infty} \frac{(-6)^n}{6^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

convergent alternating series

$x = 8$

$$\sum_{n=1}^{\infty} \frac{(6)^n}{6^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Divergent p-series

Interval of convergence $[-4, 8)$

10. (10) Find a power series representation for the function

$$f(x) = 3x^4 \arctan(x)$$

series for arctan?

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

since series & arctan(x) are both 0 when $x=0$, $C=0$ also.

$$f(x) = 3x^4 \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{3x^{2n+5}}{2n+1}$$

11. (18) This question has 6 short answer parts.

(a) Could you in principle compute $\int x^{10^{10}} e^x \, dx$, and if so, how?

yes — by parts, many times

(b) What substitution would you use to evaluate $\int x^3\sqrt{16+x^2}\,dx$?

$x = 4\tan\theta$

(c) Does $\displaystyle\int_2^\infty \frac{dx}{x^{\sqrt{2}} - \sqrt{2}}$ converge?

no, it acts like $\displaystyle\int_2^\infty \frac{dx}{x^{\sqrt{2}}}$ and $\sqrt{2} > 1$

(d) Find a formula for the general term, a_n, of the sequence $\frac{5}{2}, \frac{-8}{4}, \frac{11}{8}, \frac{-14}{16}, \ldots$

$a_n = (-1)^n \dfrac{5+3n}{2^{n+1}}$

n starting at 0

(e) Find $\lim_{n\to\infty} \frac{\ln(n)}{\ln(3n)}$.

$$= \lim_{x\to\infty} \frac{\ln x}{\ln(3x)} = \lim_{x\to\infty} \frac{1/x}{3/3x} = 1$$

$$\text{or} \quad = \lim_{x\to\infty} \frac{\ln x}{\ln x + \ln 3} = 1$$

(f) Find the error if you use s_4 (the sum of the first 4 terms) to approximate the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}.$$

$$|R_4| \le |a_5| = \frac{25}{32}$$