Mathematics 74-114 Midterm Examination
Spring 2012

This examination is a take-home test. It consists of five problems of varying difficulty and length. All problems are weighted equally. You are to do your own work and not discuss the exam with anyone. However, inform the instuctor if you believe that you have found a typo or mistake. For sources you may use the textbook or your class notes, but nothing else. You may cite a result without proof if it appears in either (1) the assigned reading in the text (2) the assigned homework (3) your class notes. Even if you cannot solve one part of a problem, you may still use that result in a later part of the problem.

Important : Write on one side of the paper and show all work. Give a reason for each of your assertions (except the obvious and well-known ones), but try to keep your solutions short. To receive full credit for your work you must explain it clearly and write it legibly.

I. Let $p : \widetilde{X} \to X$ be a covering map which is n-sheeted, $2 \leq n \leq \infty$. Prove that there is no map $s : X \to \widetilde{X}$ such that $ps = \mathrm{id}$. (Such a map is called a **section** of p.)

II. Let G be a group with unit e and let $S \subseteq G$ be a set. The **normal closure** $\overline{S}$ of S is defined to be the intersection of all normal subgroups of G which contain S. Prove

$$\overline{S} = \{e\} \cup \{c_1 \cdots c_k \mid k \geq 1, \ c_i = a_i s_i^{\epsilon_i} a_i^{-1}, \text{ where } a_i \in G, \ s_i \in S \text{ and } \epsilon_i = \pm 1\}.$$

III. For any two based spaces (U, u_0) and (V, v_0) let $[U, V]$ denote the set of based homotopy classes of based maps $(U, u_0) \to (V, v_0)$. Now let (A, a_0), (X, x_0) and (Y, y_0) be based spaces and define

$$\theta : [A, X \times Y] \to [A, X] \times [A, Y]$$

by $\theta[f] = ([p_1 f], [p_2 f])$, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections. Prove that θ is a well-defined bijection.

IV. Let $f : X \to Y$ be a map and let $p : \widetilde{Y} \to Y$ be a covering map. Define the **pull-back** P by

$$P = \{(x, \widetilde{y}) \mid x \in X,\ \widetilde{y} \in \widetilde{Y} \text{ with } f(x) = p(\widetilde{y})\}.$$

Define maps $q : P \to X$ and $r : P \to \widetilde{X}$ by $q(x, \widetilde{y}) = x$ and $r(x, \widetilde{y}) = \widetilde{y}$.

1. Prove that $q : P \to X$ is a covering map.

2. Prove that r induces a bijection $q^{-1}(x) \to p^{-1}(f(x))$.

3. Prove that there is a section for $q : P \to X$ (that is, a map $s : X \to P$ such that $qs = \text{id}$) if and only if f can be lifted to $\widetilde{Y}$.

V. Let $\widetilde{X}$ be any normal cover of X with covering map p, let $x_0 \in X$ be the base point and choose $\widetilde{x}_0 \in p^{-1}(x_0)$. Define $\theta : \pi(X, x_0) \to \mathcal{A}(\widetilde{X})$ (the group of deck transformations) as follows: Let $\alpha = [l] \in \pi(X, x_0)$ and let $\widetilde{l}$ be the lift of l to $\widetilde{X}$ starting at $\widetilde{x}_0$. Set $x_0' = \widetilde{l}(1)$. Then $p_* \pi(\widetilde{X}, \widetilde{x}_0)$ and $p_* \pi(\widetilde{X}, x_0')$ are conjugate, hence equal. Therefore there exists $\phi \in \mathcal{A}(\widetilde{X})$ with $\phi(\widetilde{x}_0) = x_0'$. Set $\theta(\alpha) = \phi$. Prove

1. θ is a homomorphism.

2. Kernel $\theta = p_* \pi(\widetilde{X}, \widetilde{x}_0)$.

Thus θ induces a homomorphism $\theta' : \pi(X, x_0)/p_* \pi(\widetilde{X}, \widetilde{x}_0) \to \mathcal{A}(\widetilde{X})$, where $\pi(X, x_0)/p_* \pi(\widetilde{X}, \widetilde{x}_0)$ is the set of right cosets. Prove

3. $\lambda \theta' = \mu$, where $\lambda : \mathcal{A}(\widetilde{X}) \to p^{-1}(x_0)$ and $\mu : \pi(X, x_0)/p_* \pi(\widetilde{X}, \widetilde{x}_0) \to p^{-1}(x_0)$ have been defined in class.