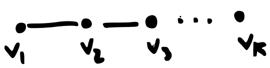
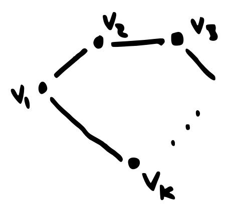
The Matrix-Tree Theorem

First, what's a tree?

A <u>path</u> in a graph G is a sequence $v_1, v_2, \ldots, v_k$ of vertices so that $v_1 \sim v_2 \sim \cdots \sim v_k$.



A <u>cycle</u> is a path $v_1, v_2, \ldots, v_k$ with $k \geq 3$ such that $v_1 \sim v_k$.



A graph is <u>connected</u> if there is a path between every two of its vertices.

A <u>tree</u> is a connected graph without cycles.

<u>Theorem</u>: The following are equivalent for a graph T:

1. T is a tree,
2. any two vertices of T are connected by a unique path,
3. T is minimally connected, i.e., $T-e$ is disconnected for every edge e of T,
4. T is maximally acyclic, i.e., $T+e$ has a cycle for every edge e not in T.

<u>Proof</u>: HW.

<u>Fact</u>: Every tree with $n \geq 2$ vertices has at least two leaves (vertices of degree 1).

<u>Proof</u>: Let T be a tree. Since T is connected, $\deg v \geq 1$ for all $v \in T$. Now

$$\sum_{v \in T} \deg v = \underbrace{2(n-1)}_{\text{each edge counted twice}} = 2n-2,$$

so $\deg v = 1$ for at least two vertices. ■

<u>Fact</u>: If G has n vertices and $n-1$ edges and is not a tree, then G has an isolated vertex (vertex of degree 0).

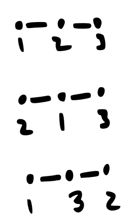
<u>Proof</u>: Similar. ■

How many trees on $[n]$ are there?

$n=2$:

2^0

$n=3$:



3^1

$n=4$: 4^2

<u>Cayley's Theorem</u>: There are n^{n-2} trees on $[n]$.

Another perspective

The graph $H=(V_H,E_H)$ is a <u>subgraph</u> of the graph $G=(V_G,E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

H is a <u>spanning subgraph</u> of G if $V_H = V_G$ and $E_H \subseteq E_G$.

H is a <u>spanning tree</u> of G if H is a spanning subgraph of G and H is a tree.

The complete graph K_n has vertices $[n]$ and all edges $\binom{[n]}{2}$.

<u>So</u>: how many spanning trees does K_n have?

We will actually figure out how to compute the number of spanning trees for any graph G.

<u>Def</u>: Let G be a directed graph without loops. Let

$$V_G = \{v_1, \dots, v_n\}$$

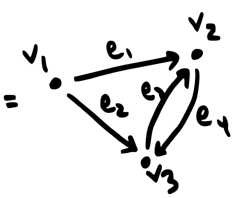
and

$$E_G = \{e_1, \dots, e_m\}.$$

The <u>incidence matrix</u> of G is the $n \times m$ matrix A defined by

- $A_{i,j} = 1$ if e_j ends at v_i
- $A_{i,j} = -1$ if e_j begins at v_i
- $A_{i,j} = 0$ otherwise.

<u>Ex</u>:

$G =$ 

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

\# spanning trees?

$e_1e_2 \quad e_1e_3 \quad e_1e_4$
$e_2e_3 \quad e_2e_4$ $= 5$

<u>Theorem 10.20</u> Let G be a directed graph without loops, and let A be the incidence matrix of G. Remove any row (corresponds to a vertex) from A to obtain the matrix A_0. The number of spanning subtrees of G is $\det A_0 A_0^T$.

<u>Note</u>: Spanning tree of a directed graph? This just means that if you ignore the directions you have a spanning tree.

__Proof:__ The Binet-Cauchy formula states that

$$\det A_0 A_0^T = \sum (\det B)^2$$

where the sum ranges over all $(n-1)\times(n-1)$ submatrices B of A_0.

(Here $n = \#$ vertices of G.)

Every such B corresponds to a spanning subgraph H of G with $n-1$ edges.

Claim: $\det B = \begin{cases} \pm 1 & \text{if } H \text{ is a tree,} \\ 0 & \text{otherwise.} \end{cases}$

We prove the claim by induction on n.

__Back to Ex:__

Remove last row of A to get

$$A_0 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

which 2×2 matrices B give $\det B \neq 0$?

$\det \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{array} = 1.$	$\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1.$
$\det \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} = -1.$	$\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = +1.$
$\det \begin{array}{c} \begin{array}{cc} 1 & 4 \end{array} \\ \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \end{array} = 1.$	$\det \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = 0.$

$\#$ spanning trees $= 5.$

First suppose that H has two leaves. Then at least one of these leaves corresponds to a row of A_0, so B has a row with a single nonzero entry. Expanding $\det B$ along this row, we see that

$$\det B = \pm \det B',$$

where B' denotes the submatrix of B with this row and corresponding column removed. B' defines a graph itself, H'. By induction, $\det B' = \pm 1$ if H' is a tree and 0 otherwise.

Now suppose that H does not have two leaves. Then H is not a tree, and also, since H has only $n-1$ edges, H must have a vertex of degree 0 (an isolated vertex). Thus B has an all-0 row, so $\det B = 0$. ■