An alternative formulation

If $x \le y$ in P, then
$$[x,y] = \{ z : x \le z \le y \}$$
is a <u>closed interval</u> in P.

We defined the incidence algebra $I(P)$ as the set of all R-valued (for some ring R) matrices M indexed by the elements of P such that $M(x,y) = 0$ unless $x \le y$.

Others define $I(P)$ as the set of all functions $f : \text{Intervals} \to R$ with multiplication given by
$$(f \cdot g)([x,y]) = \sum_{x \le z \le y} f(x,z)\, g(z,y).$$
These are equivalent.

Lattices

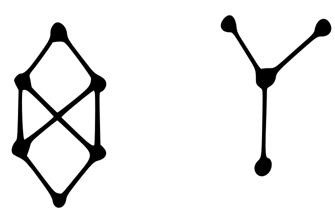
The poset P is called a <u>lattice</u> if every pair of elements $x, y \in P$ has

- a unique least upper bound, called the <u>join</u> and denoted $x \vee y$, and
- a unique greatest lower bound, called the <u>meet</u> and denoted $x \wedge y$.

<u>Ex</u>: $P = (2^{[n]}, \subseteq)$ is a lattice,
$S \vee T = S \cup T$,
$S \wedge T = S \cap T$.

<u>Ex</u>: $P = (\mathbb{N}, \mid)$ is a lattice,
$m \vee n = \mathrm{lcm}(m,n)$,
$m \wedge n = \gcd(m,n)$.

Not all posets are lattices



Minimum and Maximum elements

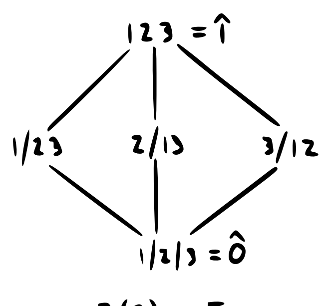
By successively taking meets, we see that every finite lattice has a unique minimum element. This element is usually denoted $\hat{0}$.

By symmetry, finite lattices have unique maximum elements, denoted by $\hat{1}$.

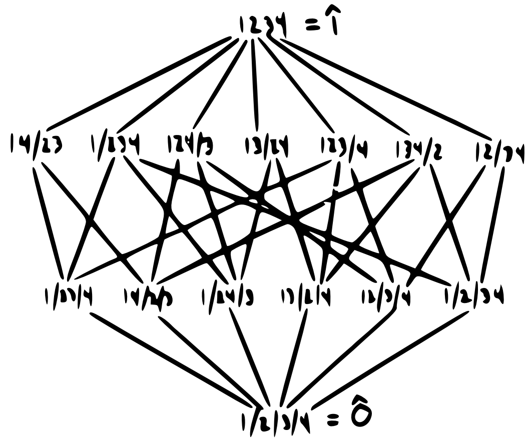
Set partitions

Let Π_n denote the collection of all (set) partitions of $[n]$.
(Recall that $|\Pi_n| = B(n)$.)
Define an order on Π_n by refinement:

$\sigma \le \pi$ if every block of σ is contained in a block of π.



$B(3) = 5$.



$B(4) = 15.$

Ex: set partitions of cards

$\hat{1}$ = all cards in same block
$\hat{0}$ = all cards in different blocks
S = blocks are the suits
C = blocks are the colors
V = blocks are the blackjack values
N = blocks are the names (ace, King, ...)

Observe:

$S \leq C$
$N \leq V$
$N \wedge S = \hat{0}$
$N \wedge C$ is all doubleton blocks
$S \vee V = \hat{1}$

Is Π_n a lattice?

$\hat{1} = 12\cdots n$ (one block)

$\hat{0} = 1/2/\cdots/n$ (all singletons)

$\pi \wedge \tau$: the elements i and j share a block in $\pi \wedge \tau$ if and only if they share a block in both π and τ.

$\pi \vee \tau$?

Π_n is a lattice

The poset P is a <u>meet-semilattice</u> if $x \wedge y$ exists for all $x, y \in P$.

Ex: Π_n is a meet-semilattice.

Lemma 16.30: Every finite meet-semilattice with a maximum element is a lattice.

Proof: Suppose L satisfies these hypotheses and take $x, y \in L$. Let $U = \{z \in L : z \geq x, y\}$. U is nonempty because $\hat{1} \in U$, and U is finite because L is finite. Therefore U has a minimal element given by

$$u_1 \wedge u_2 \wedge \cdots \wedge u_k$$

where $U = \{u_1, u_2, \ldots, u_k\}$. This is the join of x and y. ■

Weisner's Theorem 16.33: Let L be a finite lattice with $\hat{0}$ and $\hat{1}$. Then for any element $a \neq \hat{1}$ in L,

$$\sum_{x: x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0.$$

Note: The usefulness of this result is that it allows us to compute $\mu(\hat{0}, \hat{1})$, with less hassle than our previous recurrence, which was based on

$$\sum_{\hat{0} \leq x \leq \hat{1}} \mu(x, \hat{1}) = 0.$$

Proof of Weisner's Theorem

Fix $a \neq \hat{1}$. We can rewrite the sum in the theorem as

$$\sum_{x \in L} \mu(x, \hat{1}) \begin{cases} 1 & \text{if } x \wedge a = \hat{0}, \\ 0 & \text{otherwise.} \end{cases}$$

By a bit of cleverness, this is

$$\sum_{x \in L} \mu(x, \hat{1}) \sum_{y \leq x \wedge a} \mu(\hat{0}, y).$$

Note that $y \leq x \wedge a$ iff $y \leq x$ and $y \leq a$, so by reversing the order of summation, this is

$$\sum_{y \leq a} \mu(\hat{0}, y) \sum_{x \in [y, \hat{1}]} \mu(x, \hat{1}).$$

Now since $y \leq a \neq \hat{1}$, the inner sum is always 0. ■

Example 16.34 Use Weisner's Theorem to compute $\mu_{\Pi_n}(\hat{0}, \hat{1})$.

It is up to us to choose a.

Set $a = 123 \cdots (n-1) / n$.

Then $x \wedge a = \hat{0} = 1/2/\cdots/n$ if and only if all elements of $[n-1]$ lie in different blocks in x.

There are $n-1$ such partitions

$$x = 1/2/\cdots/i-1/i\,n/i+1/\cdots/n-1.$$

Now Weisner's Theorem shows

$$\sum_{x: x \wedge a = \hat{0}} \mu_{\Pi_n}(\hat{0}, \hat{1}) = 0,$$

so we have

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = -\sum_{\substack{x \in \Pi_n \\ x \neq 0 \\ x \wedge a = \hat{0}}} \mu_{\Pi_n}(x, \hat{1}).$$

For these choices of x,

$$[x, \hat{1}] \cong \Pi_{n-1},$$

so we see that

$$\mu_{\Pi_2}(\hat{0}, \hat{1}) = -1,$$
$$\mu_{\Pi_3}(\hat{0}, \hat{1}) = -2\mu_{\Pi_2}(\hat{0}, \hat{1}) = 2,$$
$$\mu_{\Pi_4}(\hat{0}, \hat{1}) = -3\mu_{\Pi_3}(\hat{0}, \hat{1}) = -6,$$

and in general,

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = -(n-1)\mu_{\Pi_{n-1}}(\hat{0}, \hat{1});$$

thus we see by induction that $\mu_{\Pi_n}(\hat{0}, \hat{1})$ is $(-1)^{n-1}(n-1)!$ ■

The full Mobiüs function for Π_n

Suppose that $\tau \leq \pi$, where
$\pi = B_1 / B_2 / \cdots / B_k$.
Then τ is formed by splitting these blocks. Suppose B_i is split into b_i blocks in τ.

Then $[\tau, \pi] \cong \Pi_{b_1} \times \Pi_{b_2} \times \cdots \times \Pi_{b_k}$,
so
$$\mu_{\Pi_n}(\tau, \pi) = (-1)^{b_1 + \cdots + b_k - k} (b_1 - 1)! \cdots (b_k - 1)!$$

Connected graphs are counted by
$$g(\hat{1}) = g(12\cdots n).$$

But f is trivial to compute:
if $\sigma = B_1 / B_2 / \cdots / B_k$, then
$$f(\sigma) = 2^{\binom{|B_1|}{2}} 2^{\binom{|B_2|}{2}} \cdots 2^{\binom{|B_k|}{2}}.$$

Therefore, we can get a formula for $g(\hat{1})$ by Mobiüs inversion.

Counting connected graphs

There are $2^{\binom{n}{2}}$ (labeled) graphs on $[n]$.
How many are connected?

Each graph G on $[n]$ determines a set partition, whose blocks are the vertices of its connected components. Call this partition $\pi(G)$.

For $\tau \in \Pi_n$, define
$$g(\tau) = \#\text{graphs with } \pi(G) = \tau.$$
For $\sigma \in \Pi_n$, define
$$f(\sigma) = \#\text{graphs with } \pi(G) \leq \sigma, = \sum_{\tau \leq \sigma} g(\tau).$$