1.2. LECTURE 2: COMPOSITIONS AND MORE ABOUT LOG-CONCAVITY

Continuing our review of basic enumerative results, today we look at compositions. A *weak composition* of n into k parts is a sequence of k nonnegative integers which sum to n. For example, $2, 0, 1, 3$ is a weak composition of 6. A *composition* of n into k parts is a sequence of k *positive* integers which sum to n. For example, $2, 1, 3$ is a weak composition of 6.

Theorem 1. There are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ weak compositions of n into k parts.

Proof. We want to count the number of sequences $c_1, c_2, \ldots, c_k$ which sum to n. We count this quantity using the "balls and walls" formulation. Express the composition $c_1, c_2, \ldots, c_k$ as a sequence of c_1 balls, followed by a wall, followed by c_2 balls, followed by another wall, and so on. For example, the composition $2, 0, 1, 3$ corresponds to

$$\bullet\bullet||\bullet|\bullet\bullet\bullet.$$

A weak composition of n into k parts will contain n balls and $k-1$ walls (one between each part). Therefore, there are as many weak compositions of n into k parts as there are ways to choose which $k-1$ of the $n+k-1$ symbols will be walls (or which n of the $n+k-1$ symbols will be balls). $\square$

We now use this approach to count compositions.

Theorem 2. For $n \geqslant 1$, there are $\binom{n-1}{k-1}$ compositions of n into k parts.

Proof. If we have a composition of n into k parts then by subtracting 1 from each part, we obtain a weak composition of $n-k$ into k parts. Therefore the number of these compositions is

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1},$$

as desired. $\square$

Note the "for $n \geqslant 1$" hypothesis in this theorem. That's there because we say that 0 has 1 composition, the *empty composition*, which has no parts.

Theorem 3. For $n \geqslant 1$, the total number of compositions of n is 2^{n-1}.

Proof. This follows from the previous result because

$$\sum \binom{n-1}{k-1} = 2^{n-1},$$

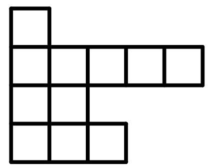

Figure 1.1: The Ferrers diagram of the composition $1, 5, 2, 3$ of 11.

but using the balls and walls approach we can give a prettier proof. In the balls and walls sequence of a composition, between any two balls there is either precisely one wall or no walls (because we are not allowed to use 0 as a part). Therefore, the number of compositions of n is equal to the number of ways to decide whether to have a wall or not between each pair of consecutive balls. Since there are n balls, there are 2^{n-1} ways to decide this. $\square$

An alternative to the balls and walls representation of a composition is its *Ferrers diagram*. The Ferrers diagram of the composition $c_1, c_2, \ldots, c_k$ consists of left-justified rows of squares in which the ith row (counting from the top down) contains c_i squares. Figure 1.1 shows an example. We say that a composition *fits inside a* $k \times m$ *rectangle* if it has at most k parts each at most m. For example, the composition in Figure 1.1 fits inside a 4×5 rectangle (or any bigger rectangle).

Now for any k and m, we can build a polynomial, $p^{k,m}$, in which the coefficient of x^n is the number of compositions of n which fit inside a $k \times m$ rectangle. Note that $p^{k,m}$ really is a polynomial, because the coefficient of x^n will be 0 for all $n \geqslant km + 1$. It is good practice to compute a few of these polynomials by hand:

$$
\begin{aligned}
p^{1,3} &= 1 + x + x^2 + x^3 \\
p^{3,1} &= 1 + x + x^2 + x^3 \\
p^{2,2} &= 1 + x + 2x^2 + 2x^3 + x^4 \\
p^{3,2} &= 1 + x + 2x^2 + 3x^3 + 4x^4 + 3x^5 + x^6
\end{aligned}
$$

First note that the constant terms are here because they count the empty composition. But more curiously, there polynomials are all unimodal! But how can we prove this? First, we develop a recurrence of the $p^{k,m}$s.

Theorem 4. If $k \geqslant 2$, then $p^{k,m} = 1 + (x + x^2 + \cdots + x^m)p^{k-1,m}$.

Proof. Consider a nonempty composition $c_1, c_2, \ldots, c_j$ of n which fits inside a $k \times m$ rectangle (so $j \leqslant k$ and $1 \leqslant c_i \leqslant m$ for all $i \in [j] = \{1, 2, \ldots, j\}$). If we remove the first part c_1, we obtain a composition of $n - c_1$ which fits inside a $k - 1 \times m$ rectangle. If the polynomial that counts these is given by

$$p^{k-1,m} = a_0 + a_1 x + \cdots + x^{(k-1)m},$$

then the number of compositions of n which fit inside a $k \times m$ rectangle is (for $n \geqslant 1$):

$$a_{n-1} + a_{n-2} + \cdots + a_{n-m},$$

and this is exactly the coefficient of n in $1 + (x + x^2 + \cdots + x^m)p^{k-1,m}$, proving the recurrence. $\square$

This recurrence expresses $p^{k,m}$ as the product of $x + x^2 + \cdots + x^m$ and $p^{k-1,m}$ (ignoring the 1 for now). We can therefore prove that the $p^{k,m}$ polynomials are unimodal using the following result of Keilson and Gerber from 1971.

> **Products of Log-Concave and Unimodal Polynomials.** Suppose that the polynomial $p(x)$ is log-concave and has positive coefficients and that the polynomial $q(x)$ is unimodal. Then their product $p(x)q(x)$ is unimodal.

Proof. Suppose that $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_rx^r$. For the purposes of the proof, it is easiest to assume that $b_0 = 0$. We may assume this without loss of generality, because if $b_0 \neq 0$, we can simply consider the polynomial $xq(x)$, which must also be unimodal.

The coefficient of x^k in $p(x)q(x)$ is

$$[x^k]p(x)q(x) = \sum_{i=0}^{k} a_i b_{k-i},$$

so the difference between the coefficient of x^{k+1} and the coefficient of x^k is

$$a_{k+1}b_0 + \sum_{i=0}^{k} a_i \left(b_{k+1-i} - b_{k-i}\right),$$

and since we have assumed that $b_0 = 0$, the $a_{k+1}b_0$ term vanishes. (Had we not have assumed that $b_0 = 0$, we would have to introduce a b_{-1} coefficient at this stage.) Now suppose that b_m is the greatest coefficient of $q(x)$, or in the case of a tie, one of the greatest coefficients. We rewrite this difference as

$$[x^{k+1}]p(x)q(x) - [x^k]p(x)q(x) = \sum_{i=0}^{k-m} a_i \left(b_{k+1-i} - b_{k-i}\right) + \sum_{i=k-m+1}^{k} a_i \left(b_{k+1-i} - b_{k-i}\right). \tag{1.1}$$

We have divided the sum into two parts to separate the positive and negative terms. The terms in the leftmost sum are, in reverse order,

$$a_{k-m}(b_{m+1} - b_m), a_{k-m-1}(b_{m+2} - b_{m+1}), \ldots, a_0(b_{k+1} - b_k),$$

so these are nonpositive, while the terms in the rightmost sum are nonnegative.

Since the sequence $\{a_k\}$ is log-concave, the ratio of consecutive terms is decreasing. Therefore,

$$\begin{aligned} a_i &\geqslant \left(\frac{a_{k-m+1}}{a_{k-m}}\right) a_{i-1} && \text{for } i \leqslant k-m\text{, while} \\ a_i &\leqslant \left(\frac{a_{k-m+1}}{a_{k-m}}\right) a_{i-1} && \text{for } i \geqslant k-m. \end{aligned}$$

Thus we may obtain an upper bound on (1.1) by replacing both a_is by $\left(\dfrac{a_{k-m+1}}{a_{k-m}}\right) a_{i-1}$:

$$\begin{aligned}
&[x^{k+1}]p(x)q(x) - [x^k]p(x)q(x) \\
&\quad \leqslant \sum_{i=0}^{k-m} \left(\frac{a_{k-m+1}}{a_{k-m}}\right) a_{i-1} \left(b_{k+1-i} - b_{k-i}\right) + \sum_{i=k-m+1}^{k} \left(\frac{a_{k-m+1}}{a_{k-m}}\right) a_{i-1} \left(b_{k+1-i} - b_{k-i}\right) \\
&\quad = \left(\frac{a_{k-m+1}}{a_{k-m}}\right) \sum_{i=0}^{k} a_{i-1}(b_{k+1-i} - b_{k-i}) \\
&\quad = \left(\frac{a_{k-m+1}}{a_{k-m}}\right) \left([x^k]p(x)q(x) - [x^{k-1}]p(x)q(x)\right).
\end{aligned}$$

Since $p(x)$ has nonnegative coefficients, $a_{k+1}/a_k \geqslant 0$. Therefore, we have just shown that once the difference in coefficients becomes negative (i.e., once the coefficients start decreasing), this difference will stay negative (i.e., the coefficients will continue decreasing), verifying that $p(x)q(x)$ is unimodal, as desired. □

This (and induction) is all we need to prove that the $p^{k,m}$ polynomials are unimodal:

Theorem 5. [Sagan (2009)] For all k and m, the polynomial $p^{k,m}$ is unimodal.

Proof. We argue by induction. For the base case, where $k = 1$, we have

$$p^{1,m} = 1 + x + x^2 + \cdots + x^m,$$

which is clearly unimodal. Now assume that $k \geqslant 2$ and that $p^{k-1,m}$ is unimodal. Our recurrence shows that

$$p^{k,m} = 1 + (x + x^2 + \cdots + x^m)p^{k-1,m}.$$

Since $x + x^2 + \cdots + x^m$ is log-concave, we know that $(x + x^2 + \cdots + x^m)p^{k-1,m}$ is unimodal. Furthermore, the coefficient of x in this polynomial is 1 (since the integer 1 has 1 composition, which fits into any nonempty rectangle), so adding a 1 to this polynomial will not violate unimodality. □