EXERCISES FOR WEEK 4

Solutions to these problems are due on Wednesday, October 21.

1. Compute

$$\mu(13/2/45/68/7, 13457/268)$$

in the set partition lattice Π_8.

2. Let P be a poset with a minimum element $\hat{0}$, and suppose that the element $x \in P$ covers only one other element, y. Prove that if $y \neq \hat{0}$, then $\mu(\hat{0}, x) = 0$.

3. Define the function $f(n)$ by

$$\sum_{d \mid n} f(d) = \log n.$$

Prove that

$$f(n) = \begin{cases} \log p & \text{if } n \text{ is a power of the prime } p, \\ 0 & \text{otherwise} \end{cases}$$

Exercises 4–6 concern the symmetric chain decomposition for $(2^{[n]}, \subseteq)$ constructed by Greene and Kleitman. Recall that to find the successor of $A \subseteq [n]$ in this SCD, we write out its characteristic vector, match 0s and 1s from left to right, and then take $A \cup \{k\}$ if k is the leftmost unmatched 0, or stop if there are no unmatched 0s. For example, to compute the successor of

$$A = \{3, 4, 6, 7, 8\} \subseteq [9],$$

we have

$$\chi(A) = 001101110,$$

so matching the 0s and 1s, we have

$$0\underbrace{\underbrace{01}1}\underbrace{01}110.$$

Thus the successor of A is $\sigma(A) = A \cup \{9\}$. Given a set A, let $U_0(A)$ denote the elements of $[n]$ which correspond to unmatched 0s, and $U_1(A)$ denote the elements of $[n]$ which correspond to unmatched 1s.

4. Prove that for any $A \subseteq [n]$, the rightmost unmatched 1 lies to the left of the leftmost unmatched 0.

5. Take $A \subseteq [n]$, and suppose that

$$\begin{aligned} U_1(A) &= \{i_1, i_2, \ldots, i_j\}, \\ U_0(A) &= \{i_{j+1}, i_{j+2}, \ldots, i_t\}, \end{aligned}$$

where $i_1 < i_2 < \cdots < i_t$. Prove that if $U_0(A) \neq \varnothing$ then

$$\begin{aligned} U_1(\sigma(A)) &= \{i_1, i_2, \ldots, i_{j+1}\}, \\ U_0(\sigma(A)) &= \{i_{j+2}, i_{j+3}, \ldots, i_t\}. \end{aligned}$$

6. Prove that this construction gives an SCD of $(2^{[n]}, \subseteq)$ by showing that if $A \subseteq [n]$ with $U_1(A) = \varnothing$ then

$$C : A \subset \sigma(A) \subset \sigma^2(A) \subset \cdots \subset \sigma^{|U_0(A)|}(A)$$

is a symmetric chain beginning at A and ending at $A \cup U_0(A)$.