EXERCISES FOR WEEK 2

Solutions to these problems are due on Wednesday, October 7.

1. Find a formula (in closed form) for $S(n, n-3)$ for all $n \geqslant 3$.

2. *Without using asymptotics for* $p(n)$, prove that $p(n)$ grows faster than any polynomial. That is, if $f(n)$ is any polynomial, prove that there is some integer N such that $p(n) > f(n)$ for all $n > N$.

3. Let $B_k(n)$ denote the number of (set) partitions of $[n]$ such that if i and j are in the same block, then $|i-j| > k$. Prove that $B_k(n) = B(n-k)$ for all $n \geqslant k$.

4. Define a family of polynomials by

$$P_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind. Use the recurrence $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ to prove that

$$P_n(x) = x\left(\frac{d}{dx}P_{n-1}(x) + P_{n-1}(x)\right).$$

5. Define the functions $Q_n(x)$ by $Q_n(x) = e^x P_n(x)$. Prove that the roots of $Q_n(x)$ are the same as the roots of $P_n(x)$ and that

$$Q_n(x) = x\frac{d}{dx}Q_{n-1}(x).$$

6. Give an inductive proof using the previous two problems that all roots of $P_n(x)$ are real. Conclude that the Stirling numbers of the second kind are log-concave.

7. Prove that in the polynomial

$$(1+x)(1+2x)\cdots(1+(n-1)x),$$

the coefficient of x^{n-k} is $c(n,k)$.