Modified Lotka-Volterra systems

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M53 15F

“Canonical” Lotka-Volterra system

- Fundamentally: Lotka-Volterra equations are a system of coupled, autonomous, 1^{st}-order ODEs adapted from individual Verhulst (logistic) models:

$$\begin{cases} \frac{dx}{dt} = x(\alpha - \beta y) \\ \frac{dy}{dt} = y(\delta x - \gamma) \end{cases}$$

with $\alpha, \beta, \gamma, \delta$ all positive constants

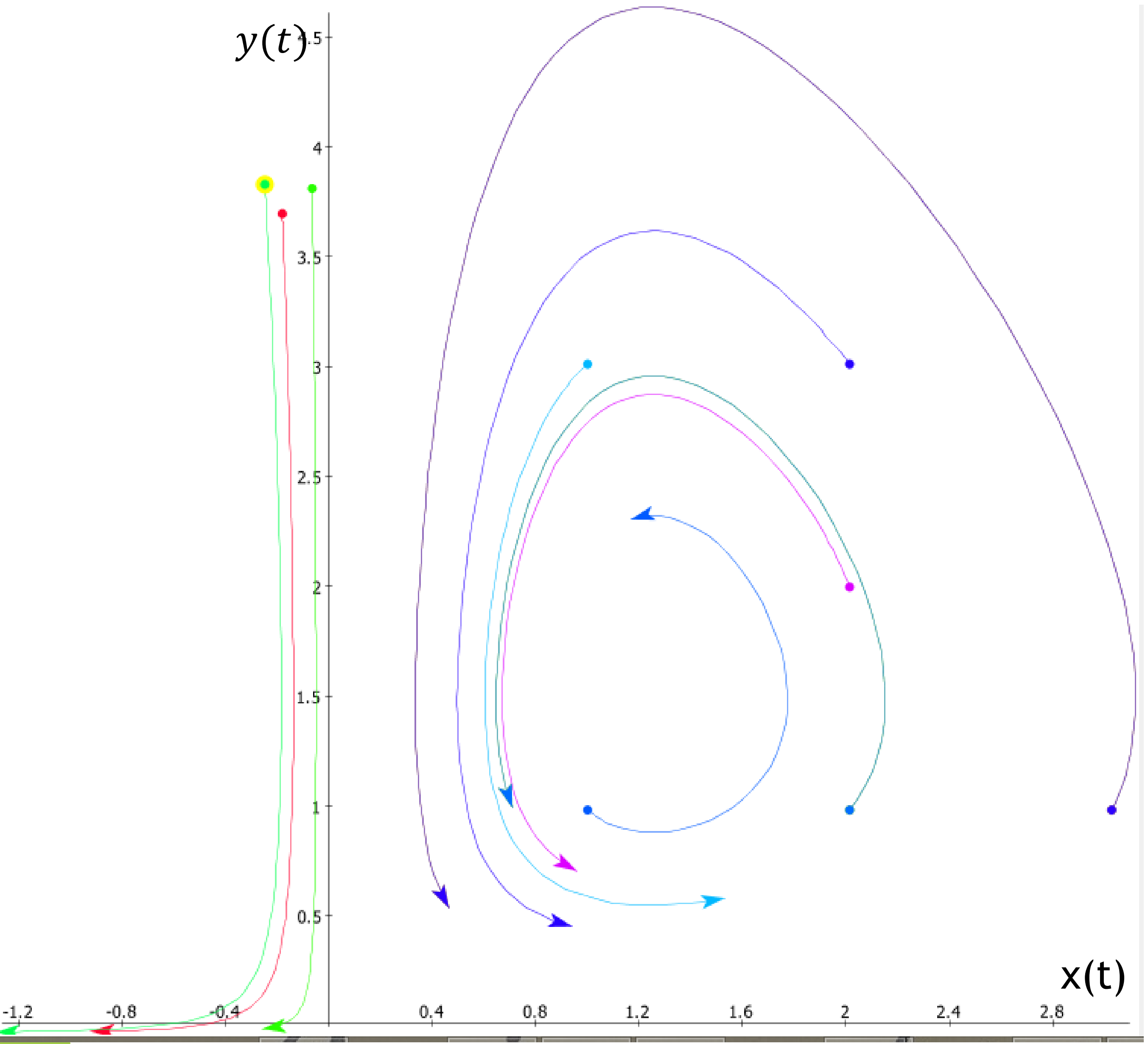
Dynamics of the canonical model:

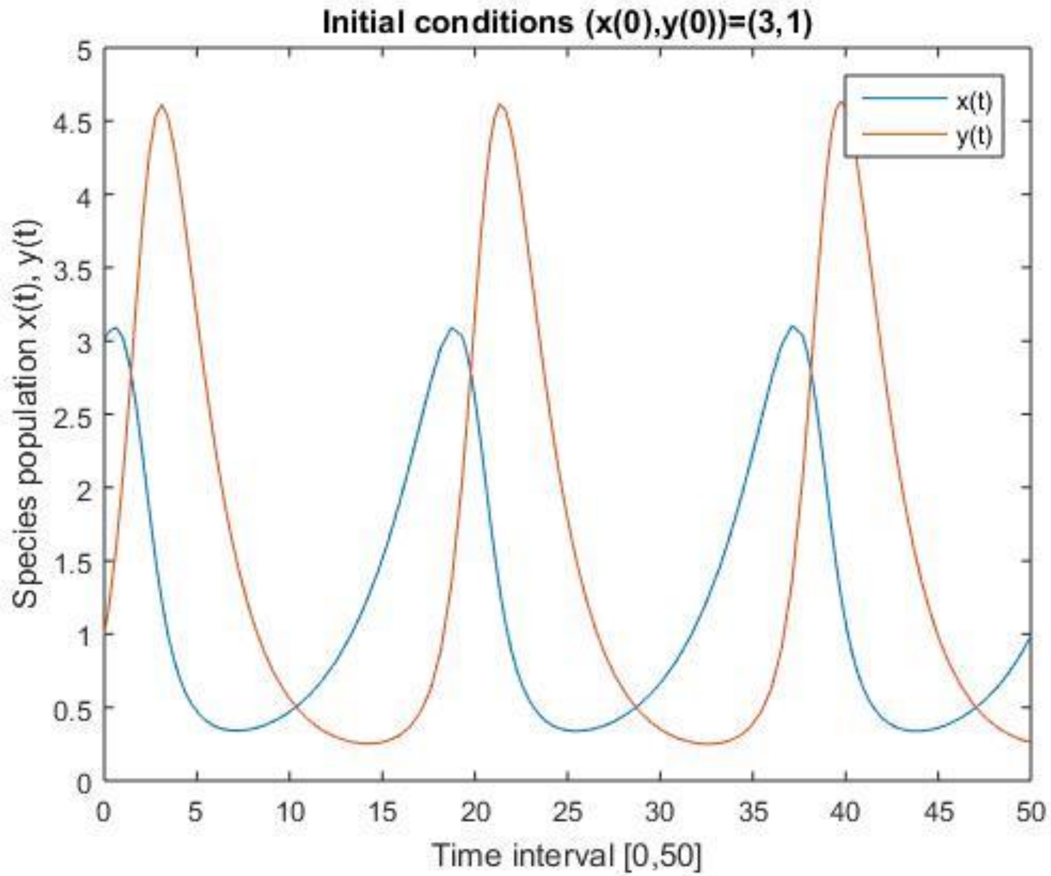
Stability: two equilibria at $(0,0)$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$

Jacobians: $J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \rightarrow$ SP , $J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix} \rightarrow \lambda = \pm i\sqrt{\alpha\gamma} \rightarrow$ stable periodic orbits

Also, as the model is autonomous and 1^{st}-order, Poincaré-Bendixson theorem predicts no chaos for these systems (even in the subsequent variations of these equations)

$$x'(t) = x(t)(0.3 - 0.2\,y(t))$$
$$y'(t) = y(t)(0.4x(t) - 0.5)$$





“Canonical” LV system with 3 species

- System: $$\left.\begin{aligned}\frac{dx}{dt} &= x(\alpha - \beta y)\\ \frac{dy}{dt} &= y(\delta x - \varepsilon z - \gamma)\\ \frac{dz}{dt} &= z(\zeta y - \eta)\end{aligned}\right\} \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \ constants$$
- α: represents the natural growth rate of in the absence of predators
- β: represents the effect of predation on x
- γ: represents the natural death rate of y
- δ: represents the efficiency rate of y in the presence of x
- ε: represents the effect of predation on species y by species z
- ζ: represents the natural death rate of the predator z in the absence of prey
- η: represents the efficiency of the predator z in the presence of prey y

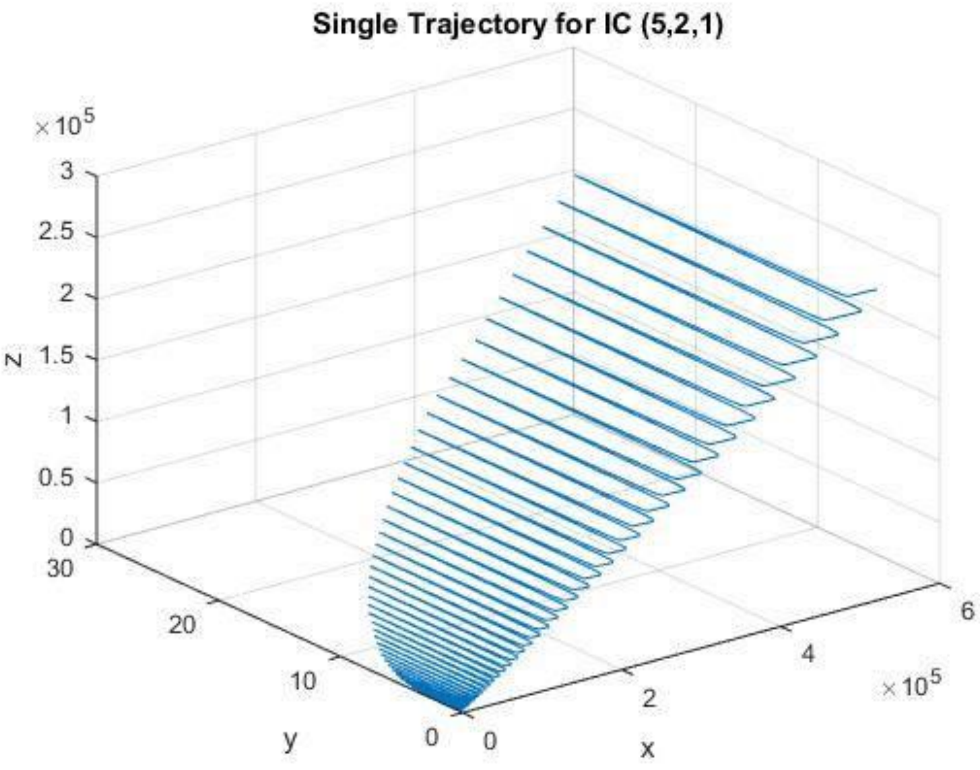
“Canonical” LV system with 3 species

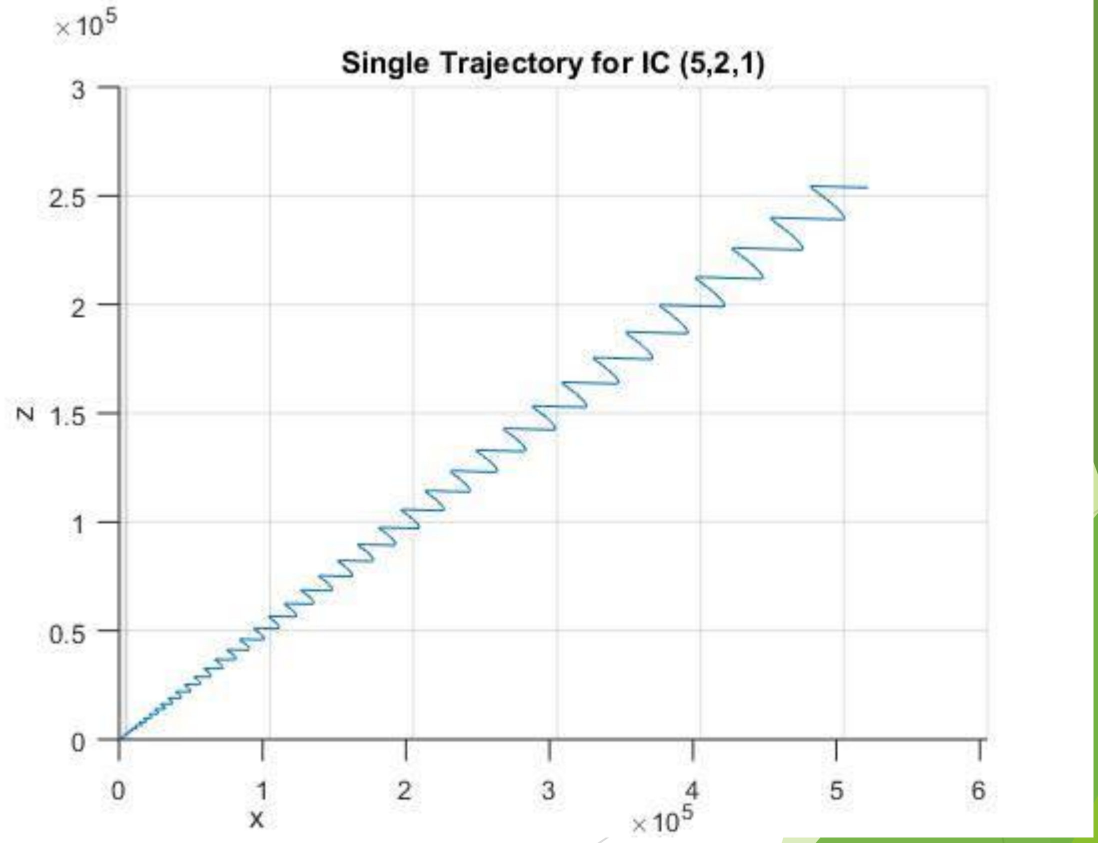
- Equilibria: 2 points: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \gamma/\delta \\ \alpha/\beta \\ 0 \end{pmatrix}$.
- Jacobean: $Df\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -x\beta & 0 \\ y\delta & \delta x - \varepsilon z - \gamma & -y\varepsilon \\ 0 & z\zeta & \zeta y - \eta \end{pmatrix}$
- For parameters α=0.3 β=0.2 γ=0.4 δ=0.5 ε=0.1 ζ=0.7 η=0.3:

For $Df\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ are $\lambda = \pm 0.3, +0.4 \rightarrow$ saddle, and

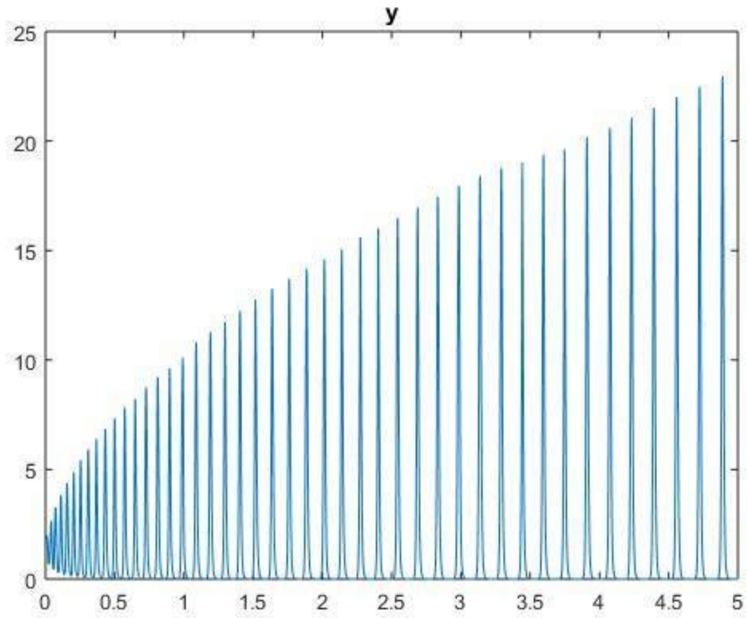
For $Df\begin{pmatrix} \gamma/\delta \\ \alpha/\beta \\ 0 \end{pmatrix}$ are $\lambda = \pm 0.1095i, 0.75$ →unstable

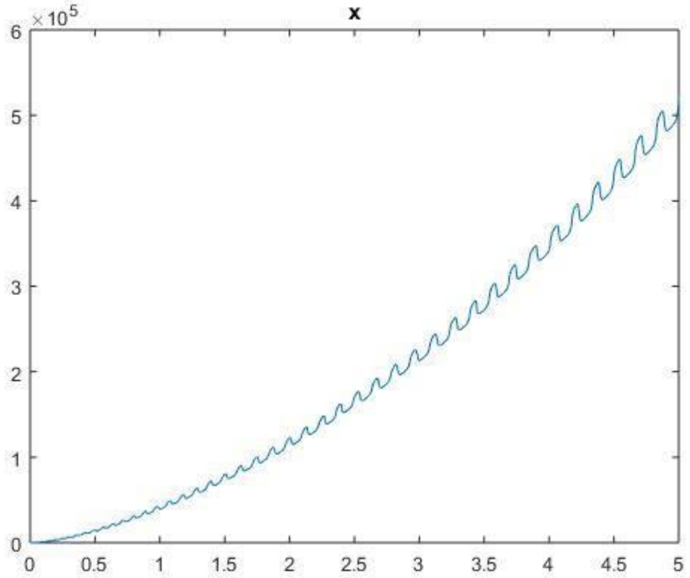
“Canonical” LV system with 3 species

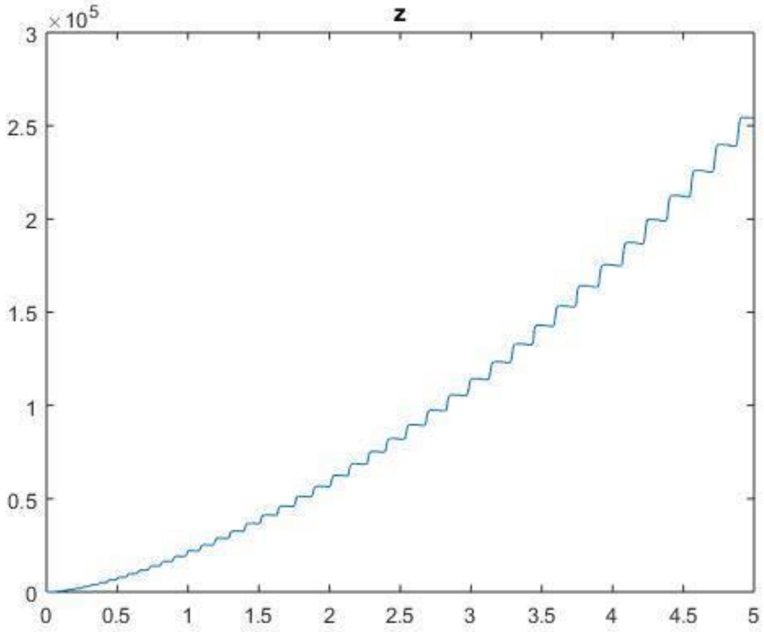




“Canonical” LV system with 3 species





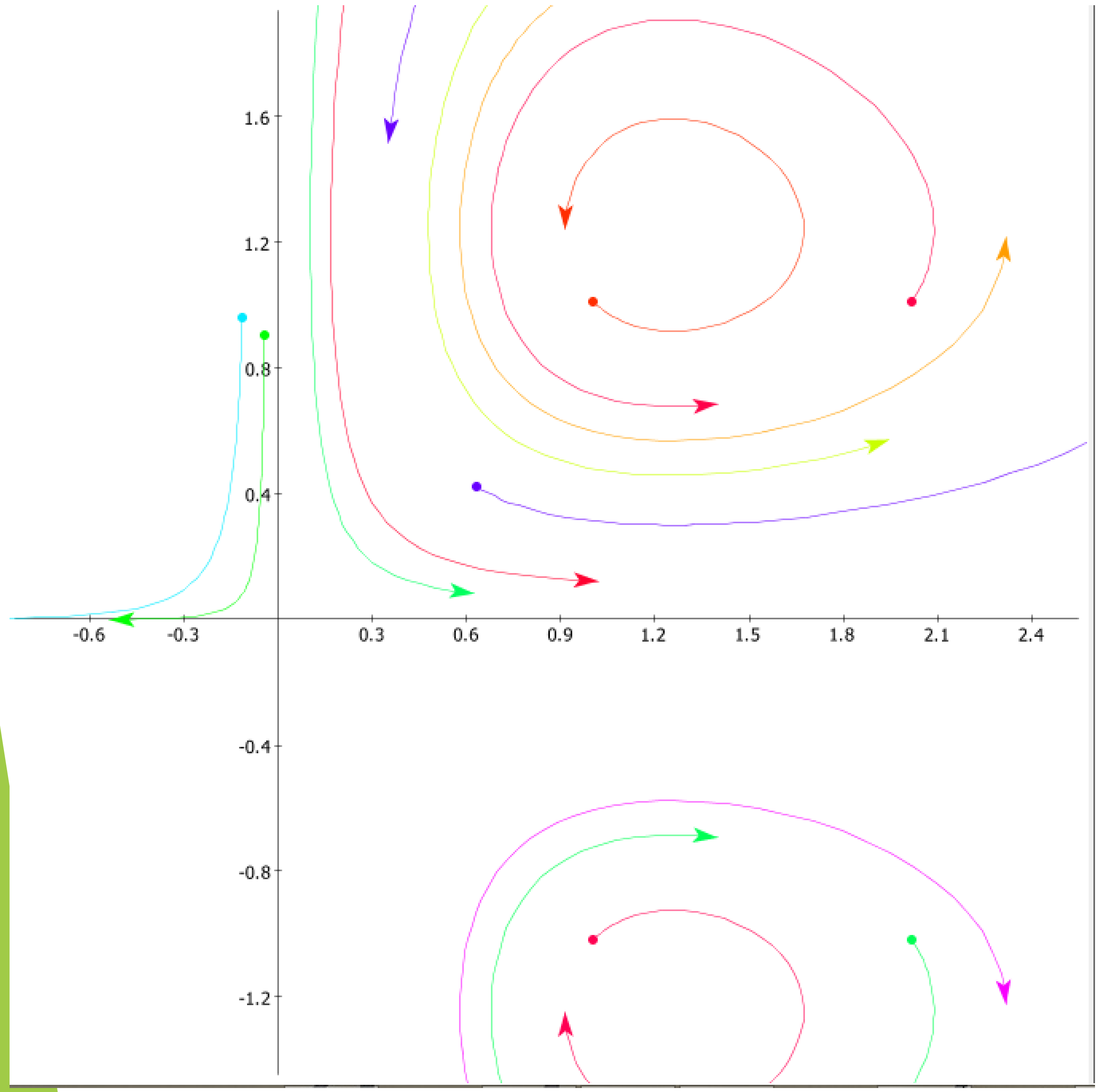


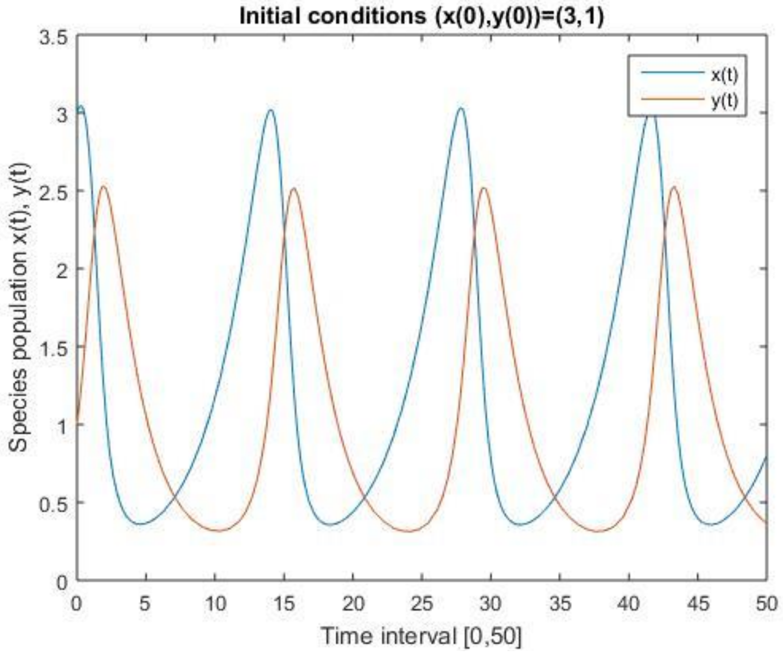
Modifying the equations (more complex predator-prey behavior)

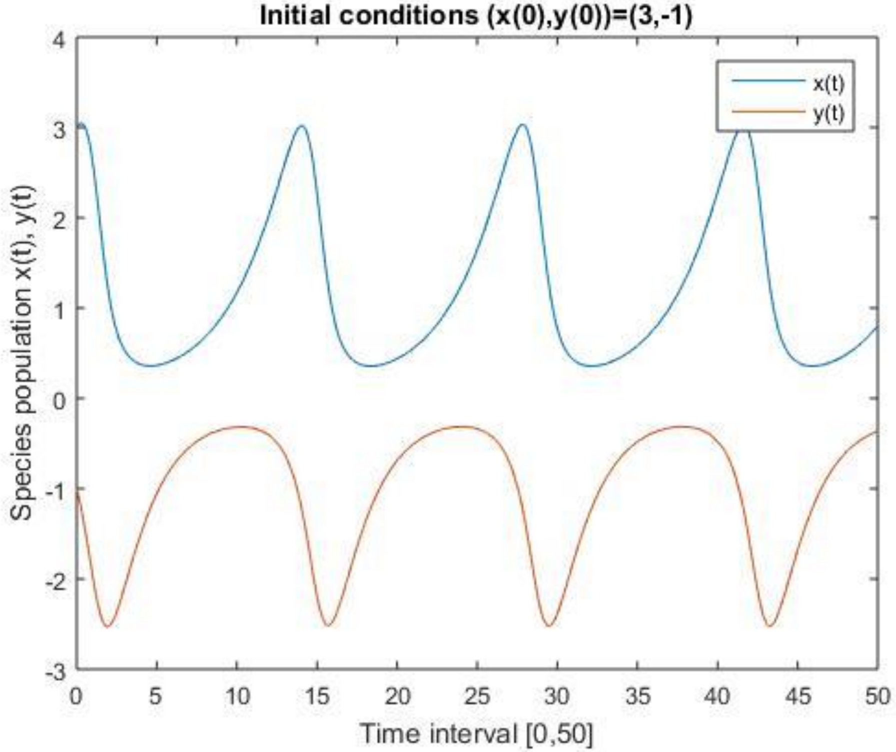
- We will look at variations to the relative consumption behavior (overconsumption of $x(t)$ by $y(t)$):
- first when $(\alpha - \beta y)$ is changed to $(\alpha - \beta y^n)$ (explicitly here, case of $n = 2$)
- ***Dynamics:***

Equilibria at $(0,0)$ ($J = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$ → SP) ,$(\frac{\gamma}{\delta}, \pm\sqrt{\frac{\alpha}{\beta}})$ ($J = \begin{pmatrix} 0 & -2\beta\left(\frac{\gamma}{\delta}\right)\sqrt{\frac{\alpha}{\beta}} \\ \delta\sqrt{\frac{\alpha}{\beta}} & 0 \end{pmatrix}$ → $\lambda \in i\mathbb{R}$ → periodic orbits)

$$\begin{cases} \frac{dx}{dt} = x(\alpha - \beta y^n) \\ \frac{dy}{dt} = y(\delta x - \gamma) \end{cases}$$


1.6
1.2
0.8
0.4
-0.6
-0.3
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4
-0.4
-0.8
-1.2


Initial conditions (x(0),y(0))=(3,1)
x(t)
y(t)
Species population x(t), y(t)
Time interval [0,50]

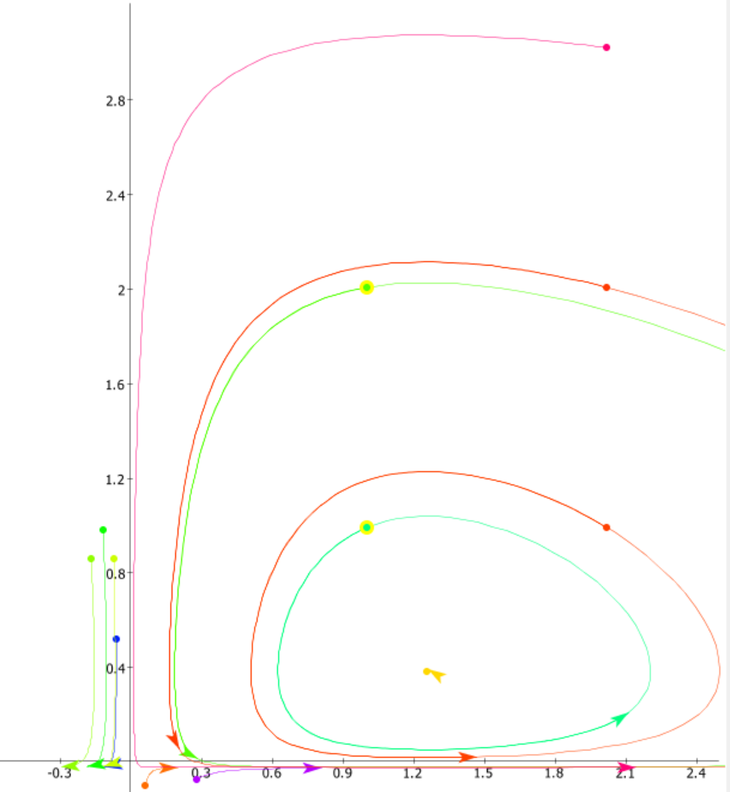

Initial conditions (x(0),y(0))=(3,-1)
x(t)
y(t)
Species population x(t), y(t)
Time interval [0,50]

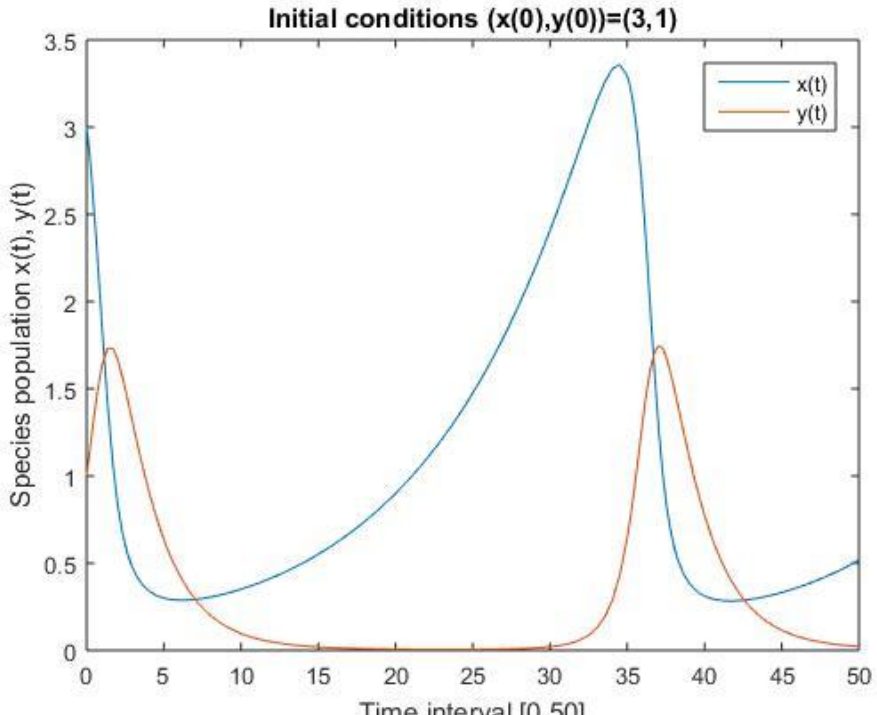
- And now changing $(\alpha - \beta y)$ to $(\alpha - \beta e^{y})$:

Equilibria at $(0,0)$ **and** $\left(\frac{\gamma}{\delta}, \log\frac{\alpha}{\beta}\right)$

For $(0{,}0)$: $J = \begin{pmatrix} \alpha-\beta & 0 \\ 0 & -\gamma \end{pmatrix}$ (then SP for $\alpha > \beta$, sink for $\alpha < \beta$)

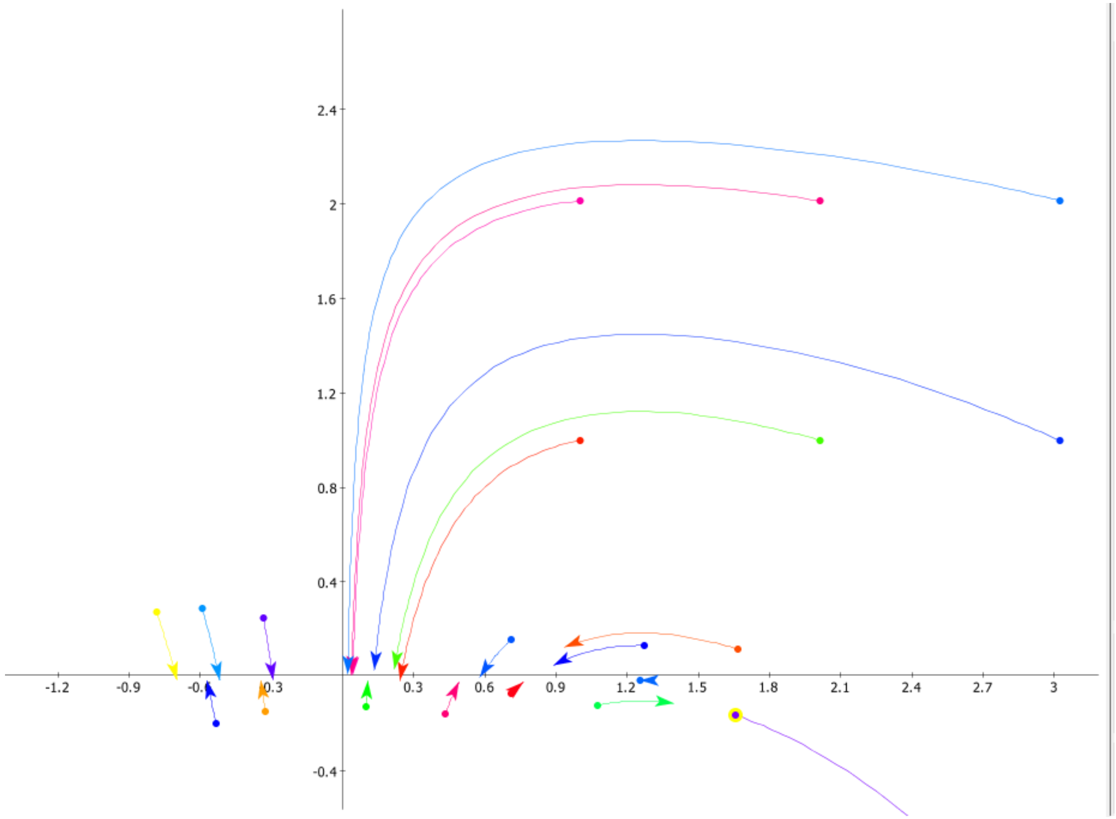
For $\left(\frac{\gamma}{\delta}, \log\frac{\alpha}{\beta}\right)$: J= $\begin{pmatrix} 0 & -\frac{\gamma\alpha}{\delta} \\ \delta\log\frac{\alpha}{\beta} & 0 \end{pmatrix}$ (periodic orbits for $\alpha > \beta$, SP for $\alpha < \beta$)

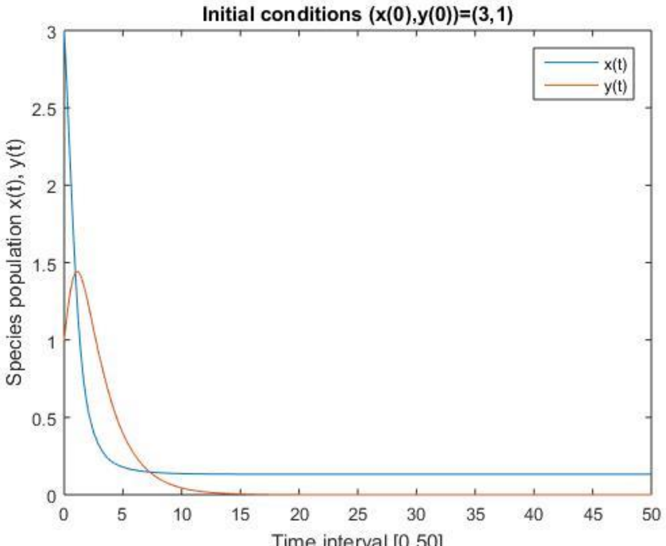




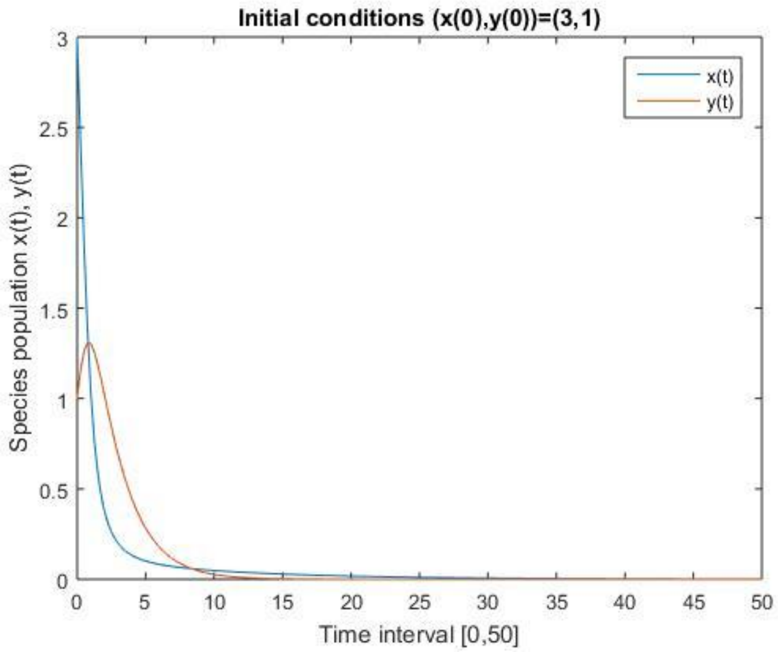
$\alpha > \beta$

$$x'(t) = x(t)(0.3 - 0.2e^{y(t)})$$
$$y'(t) = y(t)(0.4x(t) - 0.5)$$





$\alpha = \beta$



$\alpha < \beta$