Lotka-Volterra Models **Nizar Ezroura M53**

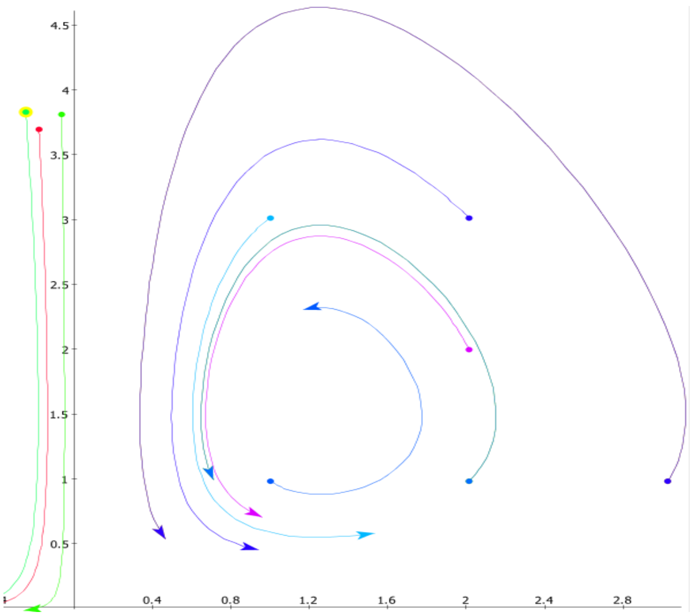
The Lotka-Volterra equations are a pair of coupled first-order ODEs that are used to describe the evolution of two elements under some mutual interaction pattern. This flexibility allows for a multitude of real-life applications in ecology (predator-prey behaviors), chemistry (reaction between two chemical species) and even economics (interacting industrial sectors). The interpretation of these equations that lends itself to the most intuitive understanding is the original one, i.e. the ecological understanding. In the initial (canonical) form, these equations can be written as: $\begin{cases} \frac{dx}{dt} = x(\alpha - \beta y) \\ \frac{dy}{dt} = y(\delta x - \gamma) \end{cases}$ with α an intrinsic growth rate for the prey species x, β an interaction term for the effect of predation on x, δ an interaction term for y and γ an intrinsic death rate for y. One assumption for this as a biological model is: y has no intrinsic growth rate, so predators only increase by consuming preys and decay otherwise.

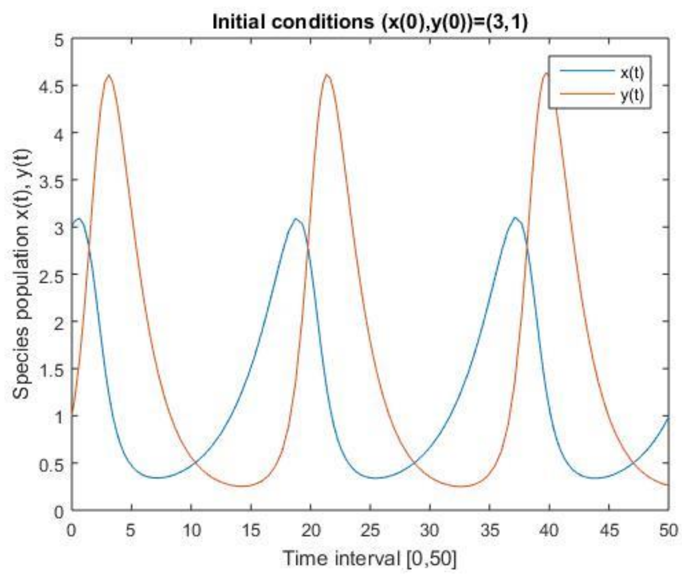
This is enough to define a dynamical system (flow) that one can formally study using flow dynamics concepts:

Equilibria: two equilibria at $(0,0)$ (a saddle point) and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ (yields pure imaginary eigenvalues). To determine the stability of this second point (which suggests periodic orbits), we need to find a Lyapunov function that is conserved and strictly positive outside of that equilibrium. $E = \delta x + \beta y - \gamma \ln x - \alpha \ln y$ (up to a constant K) satisfies these conditions as $\dot{E} = 0$ (by construction), and accepts a global minimum at that equilibrium, which means K can be chosen so that E is strictly bigger than 0 away from that point.

Individual time plots of $x(t)$ and $y(t)$ show that solutions for these are bounded and periodic, which indicates closed contour curves.

For the specific parameter values $\alpha = 0.3; \beta = 0.2; \delta = 0.4; \gamma = 0.5$ and initial condition $(x_0, y_0) = (3,1)$:



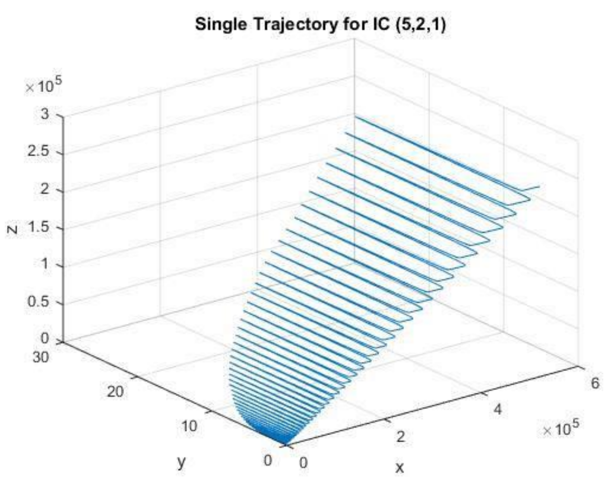


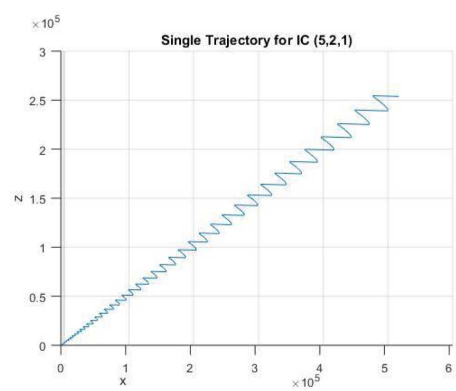
As expected, we obtain closed periodic curves around the equilibrium point$(1.25, 1.5)$. The pattern of the time plots can be understood as the biological phenomenon of prey growth until a certain maximal capacity is reached (which coincides with the predators' minimum population), and then their gradual hunting down by predators. The latter peaks again, and then decreases due to extremal absence of preys; and the cycle goes on after that.

In addition to this canonical two-species model, there is a similar system of equations set up for three interacting species. The interaction follows a directed food chain, with a prey x, an intermediary predator y and an 'apex' predator z. The system is the following:

- System: $\left.\begin{aligned} \frac{dx}{dt} &= x(\alpha - \beta y) \\ \frac{dy}{dt} &= y(\delta x - \varepsilon z - \gamma) \\ \frac{dz}{dt} &= z(\zeta y - \eta) \end{aligned}\right\}$
- α: represents the natural growth rate of in the absence of predators
- β: represents the effect of predation on x
- γ: represents the natural death rate of y
- δ: represents the efficiency rate of y in the presence of x
- ε: represents the effect of predation on species y by species z
- ζ: represents the natural death rate of the predator z in the absence of prey
- η: represents the efficiency of the predator z in the presence of prey y

In this food chain, the apex predator's population only depends on its intrinsic death rate (the model assumptions) and on the population of y. Interestingly also, the dynamics of the system yield the same equilibria as for the 2d model: $(0,0,0)$ and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}, 0\right)$ on the x-y plane (without any z-contribution). This hints at a behavior of x-y variations along parallel horizontal ($z = c$) contours, as locally z doesn't causally determine the mutual x-y behavior. When graphed:





This data also shows a clear correlative trend between x and z in addition to rapid rise-fall cycles in y. This is due to specific parameter values that lead to y varying just enough to guarantee that more x transitively translates into more z without any intermediary nuances due to y.

In bio-mathematics, many-species models are studied often for dynamical properties. One such paper [Hastings, Alan, and Thomas Powell. "Chaos in a Three-Species Food Chain."] looks at a variation on this Lotka-Volterra model that yields a more interesting phase space, with ultimately chaotic properties and the possibility for Lyapunov exponents investigations. While n-species models (with $n \geq 3$) include the possibility of chaoticity, the two-species model has an a priori determined, non-chaotic result in general. Due to the Poincaré-Bendixson theorem, bounded trajectories for autonomous systems of ODEs (basically our 2d model) can only yield as an attractor: a fixed point, a periodic orbit or a limit cycle.

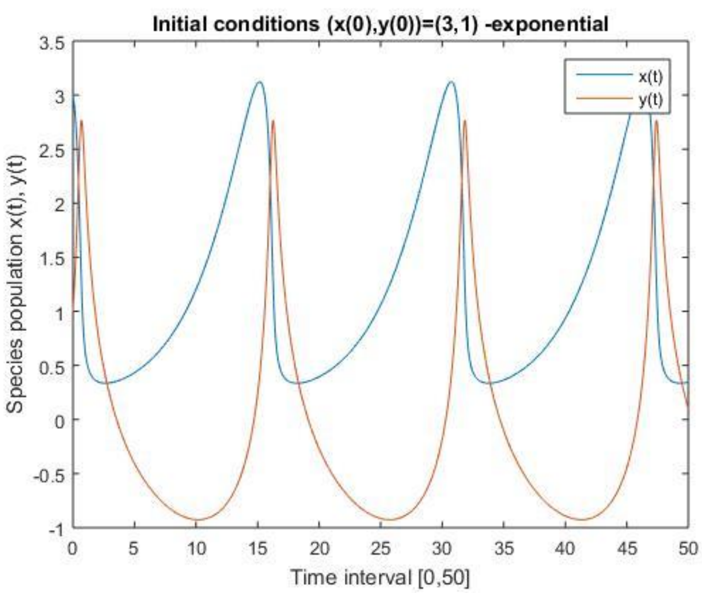
Nevertheless, the 2-species model still offers interesting population behaviors (in spite of the biological idealizations inherent to the assumptions in the model). For that reason, we have thought of investigating variations in the fundamental equations, interpreted as novel ways for the predation to happen. In particular, we looked at models of overconsumption by a few predators, which translates into y terms being turned into other monotonic functions of y such as a monomial y^n (explicitly for our case $n = 3$) and an exponential e^y. With these functions, compared to the canonical model, the same variation $x'(t)$ and $y'(t)$ is achieved with a lower number in y.

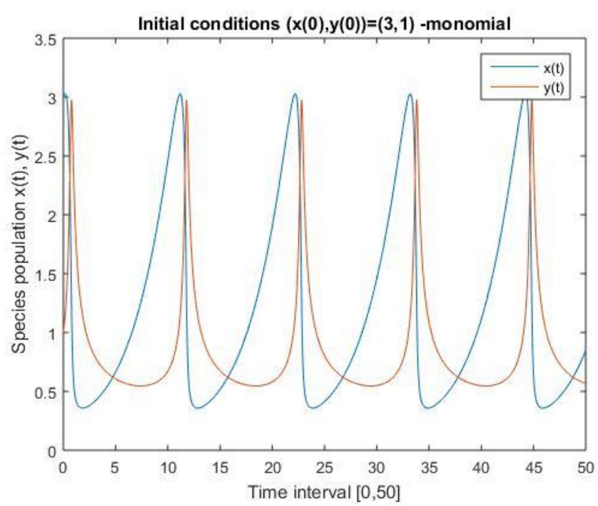
These modifications yield similar results in terms of the dynamics:

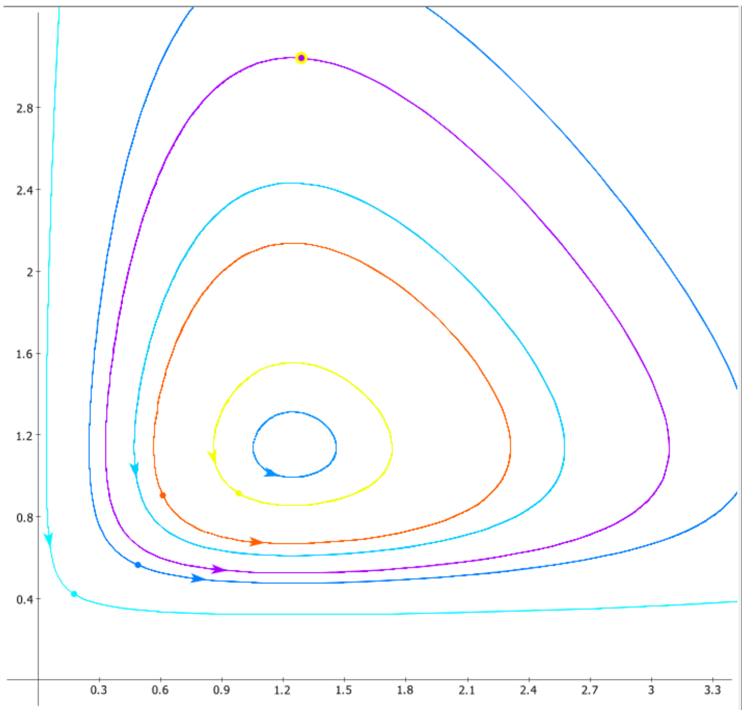
	Monomial	Exponential
Equilibria	$(0,0)$ and $\left(\frac{\gamma}{\delta}, \sqrt[3]{\frac{\alpha}{\beta}}\right)$	$\left(\frac{\gamma}{\delta}, \ln\frac{\alpha}{\beta}\right)$
Stability	$(0,0)$: saddle point $\left(\frac{\gamma}{\delta}, \sqrt[3]{\frac{\alpha}{\beta}}\right)$: purely imaginary eigenvalues	Purely imaginary eigenvalues
Lyapunov function	$E = K + \delta x + \beta y - \gamma \ln x + \frac{\alpha}{2} y^{-2}$	$E = K + \delta x + \beta y - \gamma \ln x + \alpha e^{-y}$

Both of these Lyapunov functions satisfy $\dot{E} = 0$ along contours and $E(x, y)$ being strictly positive outside of its minimum value at the equilibrium point.

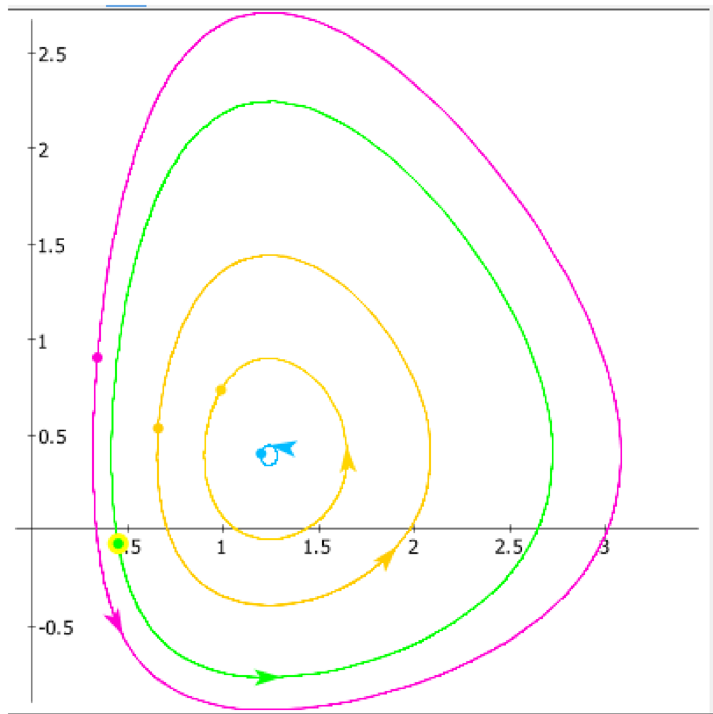
The existence of these contours leads to closed periodic orbits:







Phase space for monomial form



Phase space for exponential form. This plot presents the problematic occasional trajectories that necessarily go into negative y regions. Maintaining this model would then require using a strict ceiling to possible "energy" values E(x,y) in order to have plausible regions

On numerical accuracy:

"ode45" was used for all of the $x(t)$ and $y(t)$ computations, in solving the ODEs. A fixed parameter of relative error 10^{-5} was used, although 10^{-3} was enough to yield smooth and non-interrupted solution curves.

Appendix on MATLAB codes:

For the system solver:

```
%F = @(t, y) [ y(1)*(0.3-0.2*y(2)); y(2) *(0.4*y(1)-0.5)]
%F = @(t, y) [ y(1)*(0.3-0.2*y(2)^3); y(2)^3 *(0.4*y(1)-0.5)];
%F = @(t, y) [ y(1)*(0.3-0.2*exp(y(2))); exp(y(2)) *(0.4*y(1)-0.5)];

yo = [3;1];                                  % IC for u and v
[ts, ys]  = ode45(F, [0 50], yo,odeset('reltol',1e-5)); % numerically solve
in t domain [0,50]
figure;
plot(ts,ys(:,1))
```

```
hold on
plot(ts,ys(:,2))
xlabel('Time')
xlabel('Time interval [0,50]')
ylabel('Species population x(t), y(t)')
title('Initial conditions (x(0),y(0))=(3,1)')
legend('x(t)','y(t)')
```

For the 3-species plot:

```
function my_phase()
[~,X] = ode45(@gg,[0 5], [5 2 1]);
u = X(:,1);
w = X(:,2);
v = X(:,3);
plot3(u,w,v)
xlabel('x')
ylabel('y')
zlabel('z')
grid
title('Single Trajectory for IC (5,2,1)')
size(u)
tt=linspace(0,5,4013);
figure;
plot(tt,u)
title('x')
figure;
plot(tt,w)
title('y')
figure;
plot(tt,v)
title('z')
end

function dX = gg(t, y)
dX = zeros(3,1);
u  = y(1);
w  = y(2);
z  = y(3);
a = 5; b = 2; c = 0.3; d = 0.1; e = 0.2; f = 0.2; g = 2;
dX = [y(1)*(a-b*y(2)); y(2)*(d*y(1)-e*y(3)-c); y(3)*(g*y(2)-f)];
end
```