M53 Project

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Project Write up

Modified Lotka-Volterra Systems

Our project for Math 53 deals with Lotka-Volterra systems and their dynamical properties, with an interpretative emphasis on the biological implications of our results. For the project, Nizar Ezroura and I decided to work on the canonical (or, simply, the normally accepted) form of the Lotka-Volterra system for 2-species and for 3-species chains, and then examine two variations of the 2-species system under our personal modification.

For a 2-species chain, the simple Lotka-Volterra system is an autonomous 1st order system:

$$\frac{dx}{dt} = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = y(\delta x - \gamma)$$

Here the x variable represents the population which is being preyed on, and the y variable represents the population of the predator. The four constants, α, β, γ, δ, are used as following:

α: represents the natural growth rate of in the absence of predators

β: represents the effect of predation on x

γ: represents the natural death rate of y

δ: represents the efficiency rate of y in the presence of x

This system has two equilibrium points, $(0,0)$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. Evaluating the Jacobean at each point gives us:

$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$ Which is a triangular matrix, one eigenvalue positive one negative, so the origin is a saddle point

$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$ Whose eigenvalues are $\pm i\sqrt{\alpha\gamma}$ and therefore it has stable periodic orbits. Using the values $\alpha = 0.3, \beta = 0.2, \gamma = 0.5, \delta = 0.4$ and plotting the graph, we see:

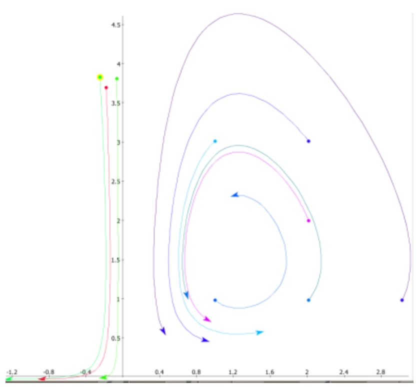


Figure 1: y(t) versus x(t)

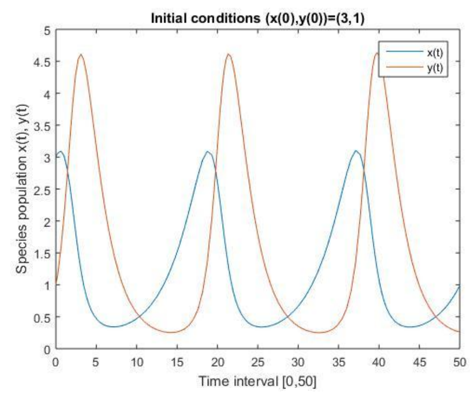


Figure 2: y(t) and x(t) versus t

In Fig. 1 we see that the origin is a saddle point, and we can also see that there exists an attracting point in the first quartile. In Fig. 2 we can see how the populations of prey and predators vary over time; we should note how they are de-phased essentially running in a circle where the minimum of one quantity approximately corresponds to the point in time for the other quantity's maximum.

Turning to the 3-species chain now, the canonical form of the Lotka-Volterra system is now:

$$\frac{dx}{dt} = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = y(\delta x - \varepsilon z - \gamma)$$
$$\frac{dz}{dt} = z(\zeta y - \eta)$$

The newly added population, *z*, is a "super" predator that predates on population *y* but does not interact directly with population *x*. Such a biological system could be a chain of rats (as *x*), snakes (as *y*), and owls (as *z*).

The meaning of the new constants bears the meaning:

ε: represents the effect of predation on species y by species z

ζ: represents the natural death rate of the predator z in the absence of prey

η: represents the efficiency of the predator z in the presence of prey

Working towards its dynamical analysis, we find the two equilibrium points to be $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\gamma/\delta\\\alpha/\beta\\0\end{pmatrix}$.

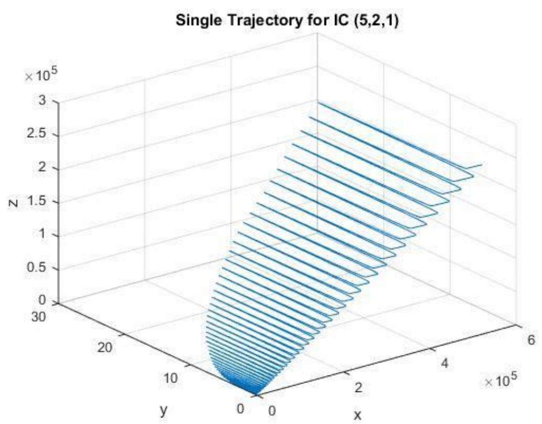
The generalized form of the Jacobean is Df $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\alpha - \beta y & -x\beta & 0\\ y\delta & \delta x - \varepsilon z - \gamma & -y\varepsilon\\ 0 & z\zeta & \zeta y - \eta\end{pmatrix}$ and at the equilibrium points:

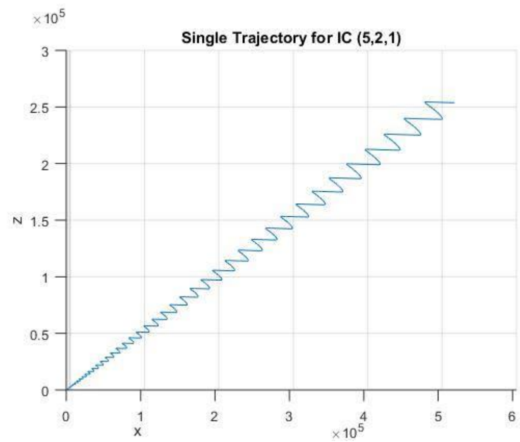
$\mathrm{Df}\begin{pmatrix}0\\0\\0\end{pmatrix}$ Has eigenvalues $\pm 0.3, \ +0.4$ so it is a saddle point.

$\mathrm{Df}\begin{pmatrix}\gamma/\delta\\ \alpha/\beta\\ 0\end{pmatrix}$ Has eigenvalues $\pm 0.1095i,\ 0.75$ so it is unstable. For the calculation of the eigenvalues we used the values for the parameters:

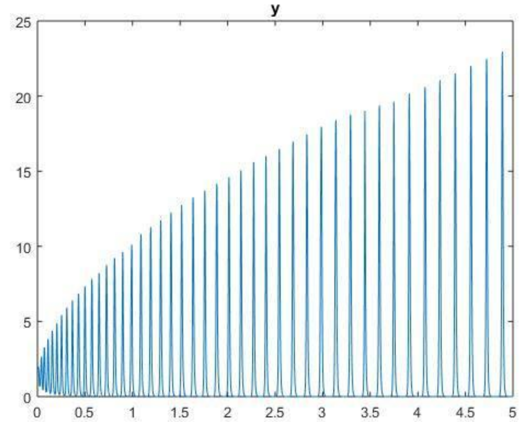
$$\alpha = 0.3\ \ \beta = 0.2\ \ \gamma = 0.4\ \ \delta = 0.5\ \ \varepsilon = 0.1\ \ \zeta = 0.7\ \ \eta = 0.3$$

The following plots show a single trajectory in three dimensions of our system:





We can see in the xz-plane plot that *x* and *z* are correlated, as expected. We cannot talk about causation in their relationship, of course, as they do not interact directly in any way, but their indirect interaction verifies that they should be positively correlated. As the population of *x* increases, there is an ambundance of resources for *y* to grow, and therefore an ambundance of resources for *z* to grow. Also, as *z* grows, its effect on the population of *y* causes the latter to be in fewer and fewer numbers, and therefore *x* can grow freely. There is a small “drawback” at every step in both graphs that we could interpret as the “phase” of cycle that we encountered in the 2-species chain example too. This drawback is slightly exaggerated in the following graph:



This figure shows *y(t)* versus t. We can see that the drawback in this case reduces it to zero after every step, or close to 0 enough so that the population would biologically not be able to recover in numbers as quickly as the model expects (certain time for members to reach maturity, for reproduction to take place, etc.) This could be expected, since there is no constant in the system that accounts for the natural growth of *y* as a population – its growth is determined totally from its predation on *x*, which is not what would realistically happen in a biosystem. But the LV system is more interested in the interaction between the species than in accurately portraying the evolution on their own.

Our personal modification of the LV system for 2-species chain had these changes:

$$\frac{dx}{dt} = x(\alpha - \beta y^n)$$
$$\frac{dy}{dt} = y(\delta x - \gamma)$$

Which has a monomial term in place of the *y* factor in the first equation, and

$$\frac{dx}{dt} = x(\alpha - \beta e^y)$$
$$\frac{dy}{dt} = y(\delta x - \gamma)$$

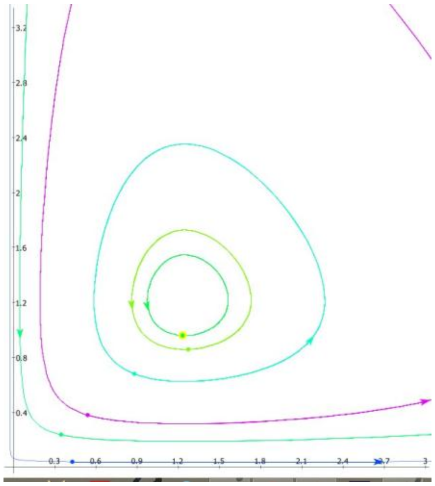
which has an exponential term. The following table summarizes our analysis for both systems:

System with monomial term for n=3	System with exponential term
Equilibria: $(0,0)$ and $\left(\frac{\gamma}{\delta}, \sqrt[3]{\frac{\alpha}{\beta}}\right)$	$(0,0)$ and $\left(\frac{\gamma}{\delta}, \log\frac{\alpha}{\beta}\right)$
Jacobeans: $J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$ $J(\frac{\gamma}{\delta}, \pm\sqrt{\frac{\alpha}{\beta}}) = \begin{pmatrix} 0 & -2\beta\left(\frac{\gamma}{\delta}\right)\sqrt{\frac{\alpha}{\beta}} \\ \delta\sqrt{\frac{\alpha}{\beta}} & 0 \end{pmatrix}$	$J(0,0) = \begin{pmatrix} \alpha-\beta & 0 \\ 0 & -\gamma \end{pmatrix}$ $J\left(\frac{\gamma}{\delta}, \log\frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\gamma\alpha}{\delta} \\ \delta\log\frac{\alpha}{\beta} & 0 \end{pmatrix}$
Stability: $(0,0)$ is a saddle point $(\frac{\gamma}{\delta}, \pm\sqrt{\frac{\alpha}{\beta}})$ leads to periodic orbits	$(0,0)$ is a saddle for $\alpha > \beta$, sink for $\alpha < \beta$ $\left(\frac{\gamma}{\delta}, \log\frac{\alpha}{\beta}\right)$ has periodic orbits for $\alpha > \beta$, and is a saddle for $\alpha < \beta$

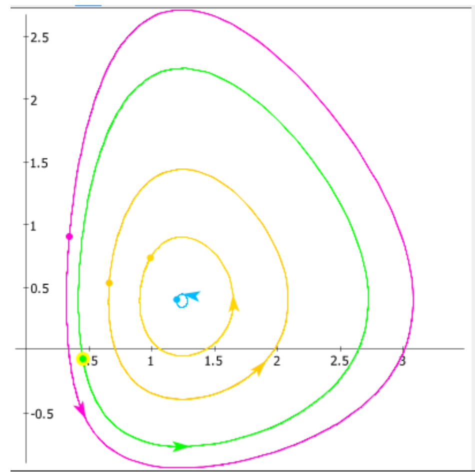
On the contours, the Lyapunov function of both systems satisfies the requirement: $\dot{E} = 0$ and $E(x, y)$ stays positive. Explicitly, the forms are:

System with monomial term for n=3	System with exponential term
$E = \delta x + ßy - \gamma\, lnx\ + \alpha/x$	$E = K + \delta x + \beta y - \gamma \ln x + \alpha e^{-y}$

Therefore both of these systems include closed periodic orbits.



Closed periodic orbits for monomial system



Closed periodic orbits for the exponential system

Biologically inaccurate, as the orbits move to negative values for the *y* population

On numerical accuracy:

"ode45" was used for all of the $x(t)$ *and* $y(t)$ *computations, in solving the ODEs. A fixed parameter of relative error* 10^{-5} *was used, although* 10^{-3} *was enough to yield smooth and non-interrupted solution curves.*

Bibliography:

Sauer, Tim, and James Yorke. "Lotka-Volterra Models." *Chaos: An Introduction to Dynamical Systems*. By Kathleen Alligood. N.p.: Springer, n.d. 309. Print

Hastings, Alan, and Thomas Powell. "Chaos in a Three-Species Food Chain." *Ecology* 72.3 (1991): 896-903. *Jstor.org*. JSTOR, 27 Mar. 2005. Web. 27 Oct. 2015.

Appendix

MATLAB codes:

For the system solver:

```
%F = @(t, y) [ y(1)*(0.3-0.2*y(2)); y(2) *(0.4*y(1)-0.5)]
%F = @(t, y) [ y(1)*(0.3-0.2*y(2)^3); y(2)^3 *(0.4*y(1)-0.5)];
%F = @(t, y) [ y(1)*(0.3-0.2*exp(y(2))); exp(y(2)) *(0.4*y(1)-0.5)];

yo = [3;1];                           % IC for u and v
[ts, ys]  = ode45(F, [0 50], yo,odeset('reltol',1e-5)); % numerically solve in t domain [0,50]
figure;
plot(ts,ys(:,1))
hold on
plot(ts,ys(:,2))
xlabel('Time')
xlabel('Time interval [0,50]')
ylabel('Species population x(t), y(t)')
title('Initial conditions (x(0),y(0))=(3,1)')
legend('x(t)','y(t)')
```

For the 3-species plot:

```
function my_phase()
[~,X] = ode45(@gg,[0 5], [5 2 1]);
u = X(:,1);
w = X(:,2);
v = X(:,3);
plot3(u,w,v)
xlabel('x')
ylabel('y')
zlabel('z')
grid
title('Single Trajectory for IC (5,2,1)')
size(u)
tt=linspace(0,5,4013);
figure;
plot(tt,u)
title('x')
figure;
plot(tt,w)
title('y')
figure;
plot(tt,v)
title('z')
end
```

```
function dX = gg(t, y)
dX = zeros(3,1);
u  = y(1);
w  = y(2);
z  = y(3);
a = 5; b = 2; c = 0.3; d = 0.1; e = 0.2; f = 0.2; g = 2;
dX = [y(1)*(a-b*y(2)); y(2)*(d*y(1)-e*y(3)-c); y(3)*(g*y(2)-f)];
end
```