Quantum Cat Map on a Torus

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1 Introduction

Many classical systems exhibit chaotic motion. The three-body problem, the double pendulum, and stadium billiards are all examples of systems which have extreme sensitivity to initial conditions. This feature of nature was discovered using the laws of classical mechanics. However, these laws of the macroscopic realm cease to apply to the behavior of microscopic systems. The dynamics of the smallest scales of nature are described by quantum mechanics. Certain quantum systems do exhibit chaotic behavior although it is different in nature than classical chaos. The transition from a classical system to its analogous quantum system is carried out through a procedure known as "quantization". The purpose of this paper is to explore the properties of a quantum system derived from the quantization of a classically chaotic system. The central question is "What chaotic features will be preserved through quantization?"

The particular classical system that we choose to investigate is one dimensional and periodic with period Δq. We further require the system to be periodic with period Δp in momentum space. Hence, the phase space (a square of dimension $\Delta p \Delta q$) has the geometry of a torus. We choose the dynamics of our system so that the time-t map is linear and hyperbolic. The specific form of the time-t map which acts on points in phase space is chosen to be the Cat Map (well known to be a chaotic).

In quantizing this toroidal classical system, we find that the available quantum states are a series of equally spaced dirac delta functions of varying complex amplitude in both the position and momentum representation. Although the classical form of this periodic system lacks a real world analogue, the quantum system actually describes the propagation of light from an ideal periodic diffraction grating. The initial state of delta functions corresponds to the the varying amplitudes of the wave at each of the slits. The subsequent time evolutions correspond to the state of the wave front at evenly spaced intervals.

We motivate the quantum dynamics of our system by first developing the analogous classical system in Section and then proceeding in Section 3 by quantizing this system. The underlying constraint of our system in both cases is periodicity in configuration space (q) and momentum space (p). In both the classical and quantum cases, this constraint will determine the allowed states for the system and restrict the form of the time-t map that generates the dynamics.

This paper closely follows the approach of Hannay-Berry [1]. The three main differences in our exploration are that we focus solely on the Cat Map, we leave out the heavy eigenvalue analysis, and we use the Husimi distribution to view the quantum dynamics more thoroughly than they do with the Wigner function (computers have come quite far since 1980!).

2 Classical Dynamics of the Cat Map

2.1 The Time-t Map

The classical dynamics of a system are governed by the Hamiltonian function $H(q, p)$. Time evolution of the dynamical variables q and p is given by derivatives of this Hamiltonian function,

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}. \tag{1}$$

We restrict our investigation to Hamiltonians which are quadratic in q and p so that the partial derivatives of H (and hence the time derivatives of our variables) are linear functions of q and p. We define H as,

$$H = aq^2 + 2bpq + cp^2. \tag{2}$$

Using Eq. 1 the continuous dynamics of our system may be organized into the vector equation,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 2b & 2c \\ -2a & -2b \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}. \tag{3}$$

We are always able to convert this relationship to a linear time-t map by transforming to an eigenbasis and integrating. The time-t map will be some matrix, T which transforms an initial state to the state a time t later,

$$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}. \tag{4}$$

Note: We have no need to relate the entries of the T matrix to the Hamiltonian coefficients because we will proceed in the next section by deriving the quantum action S from a chosen T matrix, hence foregoing the use of a particular Hamiltonian.

The form of T is restricted by Liouville's Theorem to be area preserving. In other words, the entries of T must satisfy $\det(T) = T_{11}T_{22} - T_{12}T_{21} = 1$. Furthermore, as explained by Hannay-Berry, if the map T is to preserve the periodicity of any arbitrary distribution of points in phase space then T must have integer valued entries.

2.2 Visualizing the Classical Dynamics Using Matlab

To get a sense of the classical dynamics, we take our time-t map to be given by

$$T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix},$$

which is typically called a cat map; this map will later be used to derive an analogous map in the context of quantum dynamics with similar behavior. This map is integer-valued with determinant 1, and has fixed points at $(0, 0)$ and $(0.5, 0.5)$, both of which are saddle points. In fact, this map displays chaotic behavior as well, as the following tiles suggest (Figure 1).

We began by plotting 500 random points around $(0.5, 0.5)$ and then applied our time-t map multiple times. After 4 iterations, some of these points had already been mapped around the cyclic boundary (viewing the square as a torus). After 25 iterations, it becomes pretty clear that the behavior near the equilibrium is chaotic, especially in light of the 100-iteration tile. In fact, the Lyapunov exponents of this map are $\ln(2 \pm \sqrt{3})$, and so we have chaotic behavior.

3 Quantum Dynamics of the Cat Map

The development of the quantum version of our system will come in two steps. First, we construct the form of the allowable quantum wavefunctions given our periodic constraint, then we derive the time-t map which evolves quantum states. The system is defined with periodicities Δq, Δp such that the available phase space area is an integer multiple of h, $\Delta q \Delta p = Nh$. From here on out, we change to geometrized units such that $\hbar = \frac{h}{2\pi} = 1$. Thus, the area of our phase space is $\Delta q \Delta p = 2\pi N$.

3.1 Discrete State Space

We show that the wavefunctions which exhibit periodicity in space and momentum are dirac combs. The periodicity requires $\langle q|\psi\rangle = \langle q + \Delta q|\psi\rangle$ and $\langle p|\psi\rangle = \langle p + \Delta p|\psi\rangle$, which constrains the wavefunction amplitudes as follows:

$$\begin{aligned}
\langle p|\psi\rangle &= \int dq \langle p|q\rangle\langle q|\psi\rangle \\
&= \int dq \langle p|q\rangle\langle q + \Delta q|\psi\rangle \\
&= \int dq' \langle p|q' - \Delta q\rangle\langle q'|\psi\rangle \\
&= \int dq' e^{-ip(q'-\Delta q)}\langle q'|\psi\rangle \\
&= e^{ip\Delta q} \int dq' \langle p|q'\rangle\langle q'|\psi\rangle \\
&= e^{ip\Delta q}\langle p|\psi\rangle.
\end{aligned}$$

This equation can only hold if $\langle p|\psi\rangle$ has value zero when $p\Delta q \neq 2\pi n$. In other words, $\langle p|\psi\rangle$ is a series of equally spaced Dirac delta functions,

$$\langle p|\psi\rangle = \sum_{n=-\infty}^{\infty} \phi_n \delta(p - \frac{2\pi n}{\Delta q}). \quad (5)$$

Lastly, periodicity in momentum requires the ϕ_n to repeat,

$$\begin{aligned}
\langle p|\psi\rangle &= \langle p+\Delta p|\psi\rangle \\
\sum_{n=-\infty}^{\infty} \phi_n \delta\left(p - \frac{2\pi n}{\Delta q}\right) &= \sum_{n=-\infty}^{\infty} \phi_n \delta\left(p + \Delta p - \frac{2\pi n}{\Delta q}\right) \\
&= \sum_{n=-\infty}^{\infty} \phi_n \delta\left(p - \frac{2\pi(n-N)}{\Delta q}\right) \\
&= \sum_{n=-\infty}^{\infty} \phi_{n+N} \delta\left(p - \frac{2\pi n}{\Delta q}\right) \\
\phi_n &= \phi_{n+N}.
\end{aligned}$$

With this periodicity in the discrete amplitudes ϕ_n, we may define $p_n = \frac{2\pi n}{\Delta q}$ and write the wavefunction in the momentum representation as a finite sum,

$$\langle p|\psi\rangle = \sum_{n=0}^{N-1} \phi_n \delta(p - p_n) \rightarrow |\psi\rangle = \sum_{n=0}^{N-1} \phi_n |p_n\rangle. \tag{6}$$

By reversing the roles of p and q above we can place a similar constraint on the spatial wave function,

$$\langle q|\psi\rangle = \sum_{n=0}^{N-1} \phi_n \delta(q - q_n) \rightarrow |\psi\rangle = \sum_{n=0}^{N-1} \psi_n |q_n\rangle, \tag{7}$$

where $q_n = \frac{\Delta q n}{N}$. We find that the components ϕ_n and ϕ_n are related by a discrete Fourier transform,

$$\begin{aligned}
\psi_m = \langle q_m|\psi\rangle &= \sum_{n=0}^{N-1} \phi_n \langle q_m|p_n\rangle \\
&= \sum_{n=0}^{N-1} \phi_n e^{i q_m p_n} \\
&= \sum_{n=0}^{N-1} \phi_n e^{i2\pi \frac{nm}{N}}.
\end{aligned}$$

In the transformation from classical to quantum, the nature of the physical states is drastically changed. While in the classical system the available states constitute a continuum of points in the two-dimensional phase space of q and p, in the quantum system the available states are represented by vectors in an N-dimensional Hilbert space. The square moduli of the entries of this quantum state vector in the position representation correspond to the probabilities of the particle being found at each discrete position. The momentum representation of the quantum state is given by a discrete Fourier transform of the position representation vector. Hence, the position and momentum of the particle are no longer independent quantities as they were in the classical case.

3.2 Quantum time-t map

The dynamics of a quantum state are governed by the Schroedinger equation,

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle. \tag{8}$$

This equation is integrated to obtain $|\psi(t)\rangle$,

$$|\psi(t)\rangle = e^{-i\int dtH}|\psi\rangle = U(t)|\psi\rangle. \tag{9}$$

The time-t map U is a matrix with entries $\langle q_i|U|q_j\rangle$ that each represent a transition amplitude from the discrete position q_j to q_i. One approach to obtaining the entries $\langle q_i|U|q_j\rangle$ is using the path integral method* [3]. This results in,

$$\langle q_i|U|q_j\rangle = (-1)^r \left(i\frac{\partial^2 S}{\partial q_i \partial q_j}\right)^{1/2} e^{iS(q_i,q_j)}. \tag{10}$$

Following Hannay-Berry, we assume S to be quadratic in q_i, q_j,

$$S(q_i, q_j) = \frac{1}{2}(S_{11}q_1^2 + 2S_{12}q_1q_2 + S_{22}q_2^2). \tag{11}$$

Using the classical relationships $p_1 = -\partial S/\partial q_1$ and $p_2 = \partial S/\partial q_2$, we can write relate the final and initial momenta to the final and initial positions as,

$$\begin{bmatrix} -p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \tag{12}$$

We can then relate the entries of S to the entries of T through Eq. 12 and Eq. 4 to get,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{T_{11}}{T_{12}} & -\frac{1}{T_{12}} \\ -\frac{1}{T_{12}} & \frac{T_{22}}{T_{12}} \end{bmatrix}. \tag{13}$$

We now wish to evaluate the time-t map U. The evaluation is non-trivial, however, due to the infinitely many (though periodic) dirac delta functions that constitute the quantum state in the position representation. The difficulty in computing this sum represents a divergence between the classical and quantum scenarios. In the quantum case, the careful summing is needed to keep track of the *different* phases that the quantum trajectories "pick up" as they propagate from each of the equivalent lattice sites. Hence, the wavefunction can interfere with itself. This effect does not show up in the classical case. Here, the states are represented with probability distributions that have no phase information and hence cannot cancel each other out.

In evaluating the propagator U, the amplitude of the transition $q_i \to q_f$ is the sum of transitions from the equivalent positions $q_i = Q_i/N + k$ to the positions $q_f = Q_f/N + l$, where Q_i, k, Q_f, and l are integers. Using Eq. 10, this sum is given by,

$$U_{f,i} = \sqrt{\frac{iT_{12}}{N}} \left\langle \exp\left[\frac{i\pi}{NT_{12}}\{T_{11}(Q_i + mN)^2 - 2(Q_i + mN)Q_f + T_{22}Q_f^2\}\right]\right\rangle_m,$$

with the normalization derived by the authors in [1]. These authors carry out this difficult Gauss sum, from which we plug in the Cat Map entries of T,

$$U_{f,i} = \sqrt{\frac{i}{N}} \exp\left[\frac{i2\pi}{N}\{Q_i^2 - Q_iQ_f + Q_f^2\}\right]. \tag{14}$$

This is the time-t map which we iteratively apply to our initial quantum state $|\psi_0\rangle$ to obtain dynamics. The eigenvalues and eigenvectors of this operator give insights into the behavior of the quantum dynamics. From the Schroedinger equation, we recognize the eigenvalues of U with the phases $e^{-i\omega t}$, were ω is the energy in geometrized units. The eigenstates are the states which remain fixed in time while accruing a phase factor $e^{-i\omega t}$ in each time step. Since the entries of U are of the form $e^{i2\pi r}$, with r being rational, the eigenvalues themselves must be of this same form. With this in hand we immediately see that $U^n = I$ for some finite integer n. The with the rational numbers of the eigenvalues r_1, r_2,...,r_N, we associate n with the least common multiple of their denominators,

$$U^n = \begin{bmatrix} e^{i2\pi r_1 n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i2\pi r_N n} \end{bmatrix} = \begin{bmatrix} e^{i2\pi R_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i2\pi R_N} \end{bmatrix} = I,$$

where the R_i are integers. This feature of the quantum time-t map is one of the major differences between the quantum and classical scenarios. The eigenstates of U, as seen in Section 3.4, seem to have somewhat chaotic forms. Eigenstates give a rich look into the chaos inherent in quantum systems. We do not give too much attention to them, but much analysis is applied to them in references such as [2],[4].

3.3 Husimi Distribution

Now that we have established the dynamics of our quantum system via the time-t map U, we want to iteratively apply U to initial states and *observe* the evolution. In the classical case we were able to observe the evolution of a state as a point in phase space subject to the repeated application of a linear time-t map. In the quantum case it is not as straight forward to observe the dynamics in a "phase space". The problem is that in general, quantum states cannot be represented as a probability distribution in phase space, let alone a single point with a definite position and momentum. The latter fact is easily explained by the Heisenberg uncertainty principle. The former is more subtle. The probability density $\rho(q, p)$ in phase space is used to give the probability $\rho dqdp$ that the particle has position between q and $q+dq$, and momentum between p and $p+dp$. In quantum mechanics it is meaningless to assign a probability to this "outcome" because the particle cannot have such a well defined position and momentum (again, by the Heisenberg uncertainty principle).

Although the concept of classical phase space does not translate directly to the quantum scenario, there are several analogues of phase space which allow us to visualize the quantum dynamics so as to compare and contrast them with the classical ones. The phase space analogue which we use to visualize our dynamics is the Husimi distribution. Simply put, the Husimi distribution of a state $|\psi\rangle$ is a probability distribution on phase space in which the value at (q,p) is given by the inner product of $|\psi\rangle$ with the coherent state $|\alpha(q,p)\rangle$. Coherent states are quantum states which closely resemble classical classical states. First, their uncertainty is minimal (saturating the Heisenberg inequality) and equally balanced between position and momentum. Second, they maintain this minimal and balanced uncertainty as they evolve in time. Using traveling Gaussian coherent states, the Husimi distribution defined on our state of discrete amplitudes is,

$$H(\bar{q},\bar{p}) = |\langle\alpha(\bar{q},\bar{p})|\psi\rangle|^2 = \left|\int dq\langle\alpha|q\rangle\langle q|\psi\rangle\right|^2 = \left|\sum_n \psi_n e^{-\frac{(q_n-\bar{q})^2}{2} - i\bar{p}q_n}\right|^2. \quad (15)$$

We keep in mind the fact that the subtraction $q_n - \bar{q}$ is mod(1) to conform with the periodicity of our system. Hence the gaussian nature of the coherent state is wrapped around the torus. The region of "overlap" between the left tail of the Gaussian with its own right tail (due to the periodicity) is not problematic since the values there have already diminished to a negligible amount. Furthermore, we note that the normalization of the Gaussian is ignored because we are only interested in the relative intensities in the Husimi distribution.

The "master equation" which we use to obtain the time evolution of the Husimi distribution is,

$$H(\bar{q},\bar{p},t_m) = |\langle\alpha(\bar{q},\bar{p})|U^m|\psi_0\rangle|^2. \quad (16)$$

For the purposes of implementing the Husimi distribution as used in the next section, we introduce two conceptual modifications. First, we recognize that by setting $\bar{p} = 0$, the Husimi distribution becomes the square of the convolution of our wavefunction ψ_n with a Gaussian distribution. So, $H(\bar{q}, 0)$ is the square of the state convolved with a Gaussian function. Then, we recognize the $e^{-i\bar{p}q_n}$ factor as inducing a fourier transform. Thus, we can think of the Husimi distribution as the square of the fourier transform (introduces $\bar{p}$) of the convolution (introduces $\bar{q}$) of the state $|\psi\rangle$. In the following section we refer to the convolution as the "windowing" and the Gaussian as the "windowing function".

3.4 Visualizing the Quantum Dynamics Using Matlab

To visualize the Husimi distribution, we begin with an initial state ψ_0 in our N-dimensional Hilbert space (Figure 2), and act it upon U, our quantum time-t map, i.e. $\psi_1 = U\psi_0$. For each ψ_i, we apply a Gaussian windowing function on the torus centered at each coordinate (Figure 3) thereby emphasizing the value at that coordinate and the other values centered around it. Each of the resulting windows then contain configuration information about the overlap between ψ and the coherent states described above. We then recover the momentum information by performing a discrete Fourier transformation on each of these windows. The resulting states in the transformation domain are then temporarily stored in an $N \times N$ complex-valued matrix, denoted $\mathcal{M}$. Once we've windowed each coordinate, the modulus of each component of $\mathcal{M}$ is $\sqrt{H(q,p)}$. We plot $\sqrt{H}$ as a greyscale function of phase space as seen in the image sequence of Figure 4.

The unitary matrix U that we are using can be diagonalized and its eigenvectors can be recovered. Being able to visualize quantum states, we are, in effect, able to visualize these eigenstates by extracting the eigenbasis from U. When visualized, these images appear invariant as expected, yet their values rotate about a phase equal to their associated eigenvalues in complex space. The images in Figure 5 are three examples of eigenstates, chosen with $N = 441$.

As another treat, the dynamics appear to be smooth and continuous when mapped onto a torus as seen in Figure 6.

4 Comparison of the Classical and Quantum Dynamics

The most apparent difference between these two dynamics is that the quantum evolution is periodic in time, while the evolution of the classical system leads to chaos. However, in the classical limit of the quantum system ($N \to \infty$), the period tends to infinity and we expect the chaotic classical dynamics to be restored.

Even without taking the classical limit, the quantum dynamics does retain some features of the classical dynamics. In Figure 7 we present a side by side comparison of the quantum and classical evolutions for several early time steps. The divergence in behavior between the two cases becomes apparent only after the quantum trajectories begin to overlap with one another. When this interference is viewed in the Husimi distribution we can observe two localized probability densities either cancel each other out or add together. These effects are known as destructive and constructive interference, respectively, and are determined by the relative phases of the overlapping wavefunctions (even if a single wavefunction is overlapping with itself!). In the classical case, the individual trajectories are completely independent of each other and there is no such interference. So, the orbit of each trajectory is not affected by the orbits of other points. Since there are no interference effects in the classical case, we expect interference to become negligible when taking the quantum system to the classical limit. As N goes to infinity, the number of available positions also goes to infinity. So as a Gaussian wavepacket (which has a smaller and smaller spread as $N \to \infty$) evolves in time, its overlap with itself will become far more scarce.

This investigation has shown us that the quantization of a classically chaotic system does not always lead to similar chaos in the quantum dynamics. A further investigation would explore the scaling properties of the system's dynamics with N. Lastly, the computational findings deserve to be compared to the real world quantum system that is the diffraction grating.

References

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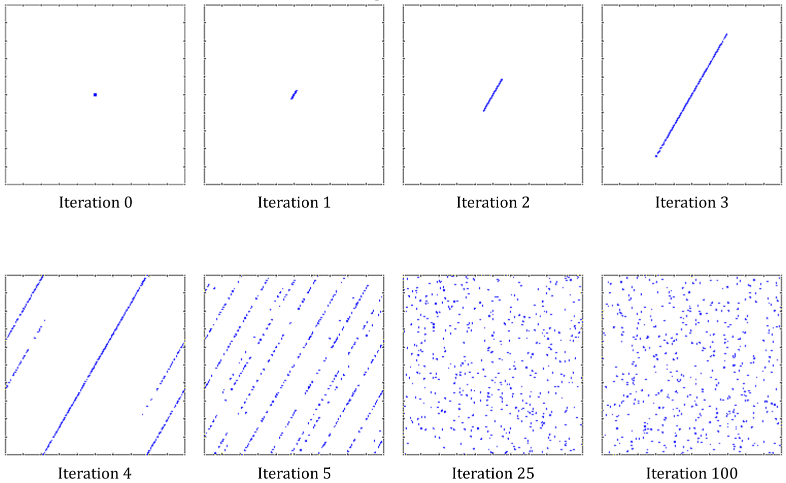
[2] S. Nonnenmacher, Anatomy of quantum chaotic eigenstates, Seminaire Poincare XIV 177 - 220, (2010).

[3] A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press (2010).

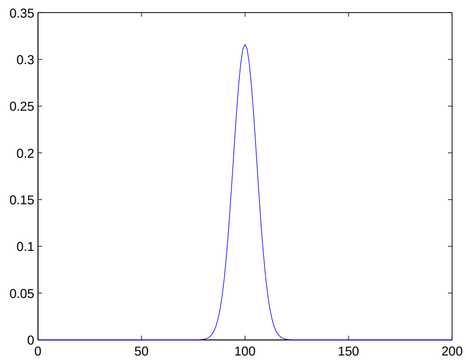
[4] A.H. Barnett, T. Betcke, CHAOS 17, 043125 (2007).

Figure 1



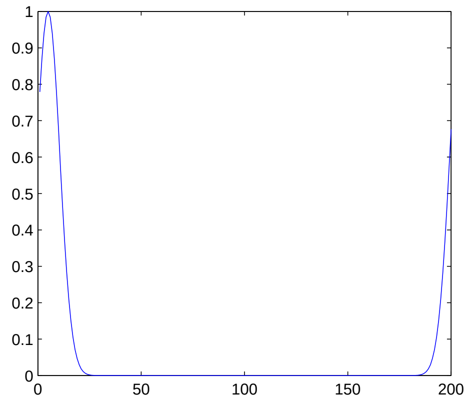
Classical Map Evolution Near Saddle Point

Figure 2



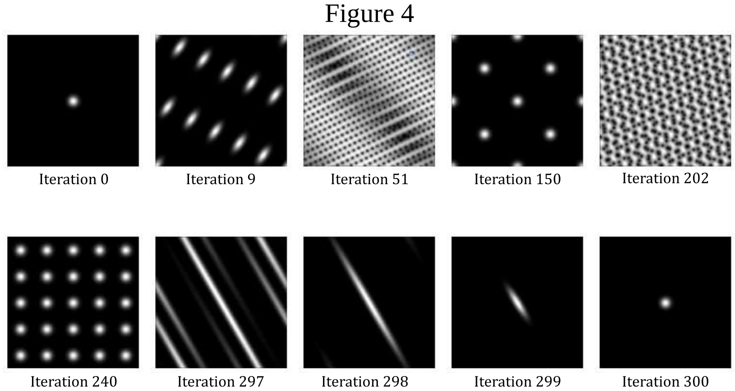
Initial Gaussian State

Figure 3



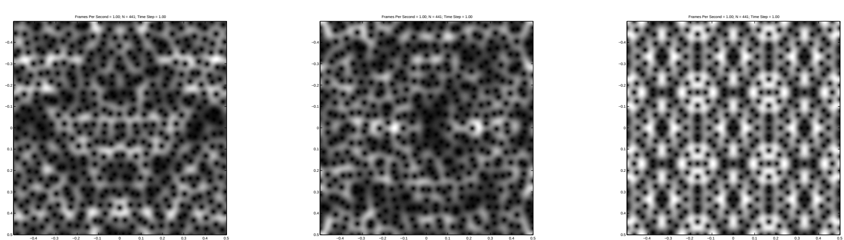
“Wrapped” Windowing Function on the Torus

Figure 4



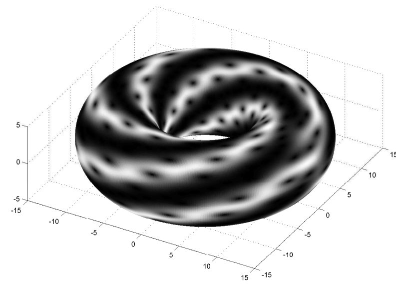
Husimi Representation of the Time Evolution of Initial Gaussian State

Figure 5



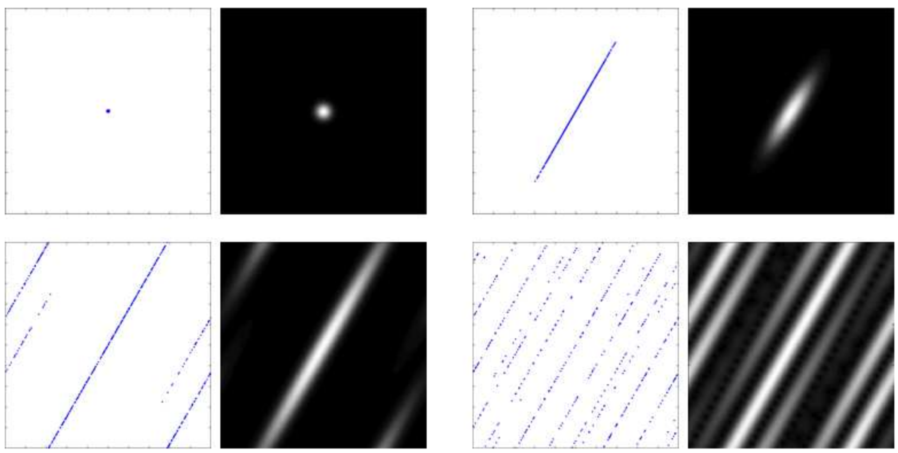
Husimi Representation of Eigenstates, $N = 441$

Figure 6



The Husimi Representation on a Torus

Figure 7



Side-by-side comparison of Classical and Quantum Evolution