The Homoclinic Tangle and Smale's Horseshoe Map

Matt Harding

I. Introduction

The homoclinic tangle is an interesting property of the manifolds of certain maps. In order to give more formal meaning to some intuitive claims about the homoclinic tangle, notably its chaotic orbits, it is useful to study the Smale Horseshoe model. Using this formulation, we can connect homoclinic orbits to a model where we can more easily define chaotic orbits.

The beginning of this paper will define the homoclinic tangle and show a numerical approximation of crossing manifolds, using MATLAB. The paper will then introduce the formulation of the Smale Horseshoe map and list some interesting properties. The paper will conclude by connecting the homoclinic tangle with this horseshoe model (with the visual aid of MATLAB), thus justifying our intuitive claim of chaotic orbits.

II. Homoclinic Tangle: Overview

Let f be an invertible map of $\mathbf{R^n}$, and let p be a fixed point saddle. Then a **homoclinic point** p' is a point that lies on both the stable and unstable manifolds of p.
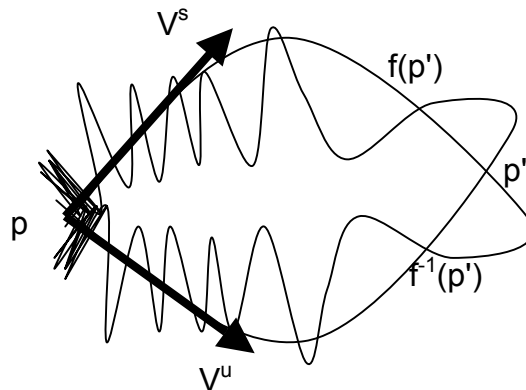
Since p' is on the stable manifold,
$$f^n(p') \rightarrow p \text{ as } n \rightarrow \infty.$$

Since p' is on the unstable manifold,
$$f^n(p') \rightarrow p \text{ as } n \rightarrow \infty.$$

Then also
$$f^n(f(p')) \rightarrow p \text{ and } f^n(f(p')) \rightarrow p \text{ as } n \rightarrow \infty,$$
$$\text{so } f(p') \text{ is a homoclinic point.}$$

Also,
$$f^n(f^1(p')) \rightarrow p \text{ and } f^n(f^1(p')) \rightarrow p \text{ as } n \rightarrow \infty,$$
$$\text{so } f^1(p') \text{ is a homoclinic point.}$$

Thus, all homoclinic points map to homoclinic points under f and f^{-1}. We call the orbits of such points **homoclinic orbits**.

We've already seen examples where the stable and unstable manifolds overlap. For example, the undamped Duffing equation $x'' - x + x^3 = 0$:
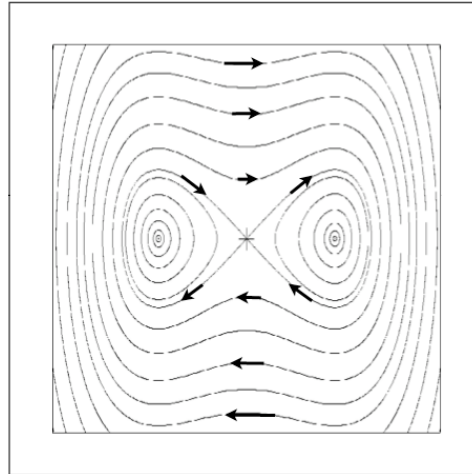
If, on the other hand, the stable and unstable manifolds do not overlap but cross transversally, then they must cross over each other an infinite number of times. This creates a complex structure called the **homoclinic tangle**. In order to study this behavior, consider the following Hénon map from Example 10.5 in *Chaos*:

$$\mathbf{f}(x,y) = (2.12 - x^2 - 0.3y, x)$$
$$\text{Fixed point } p \approx (0.945, 0.945)$$

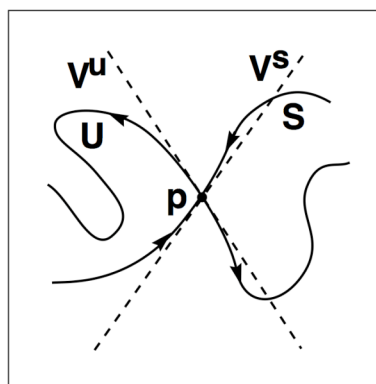$$\text{Eigenvalues of } \mathbf{Df}(p):$$

$$s \approx -0.18 \qquad \rightarrow \qquad |s| < 1$$
$$u \approx -1.71 \qquad \rightarrow \qquad |u| > 1$$

In order to approximate the stable and unstable manifolds, we cite the **Stable Manifold Theorem:**

Let f be a diffeomorphism (a smooth one-to-one map with a smooth inverse) of $\mathbf{R}^2$. Then the stable manifold S and unstable manifold U of p are tangent to the eigenvectors V^s and V^u, respectively.


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To approximate the stable and unstable manifolds, we use the technique of You, Kostelich and Yorke in "Calculating Stable and Unstable Manifolds,"[1] which uses as background the Stable Manifold theorem. We pick a point **a** along V^u within ε of p. Then we consider line segment **ab**, where **b**=f(**a**). We apply f to this line segment, and then iteratively apply f to all resulting line segments. To approximate the stable manifold, we repeat the process with V^s and f^{-1}.

The MATLAB implementation of this code, which I created, can be seen at the end of this paper (in *project.m*). The resulting plot can be seen below at left, compared to the full plot at right[2]. The light blue curve represents the stable manifold and the dark blue curve represents the unstable manifold. The fixed point is circled in both plots.

The approximated plot at left does not plot any lines, but just points. This is because the sparsely-generated iterations of f would incorrectly connect points across the gap in the curve. This plot in fact includes one million points along each iterated line segment; however, the points become very spread out upon many iterations of f^{-1}. Many of the plotted points end up far outside the plotted region. One could construct an algorithm that eliminates such points, or that provides more precision along the pieces of curves that are stretched the most. However, as I wished to focus on the theory of the homoclinic tangle and the horseshoe map for this paper, I did not construct such an algorithm.
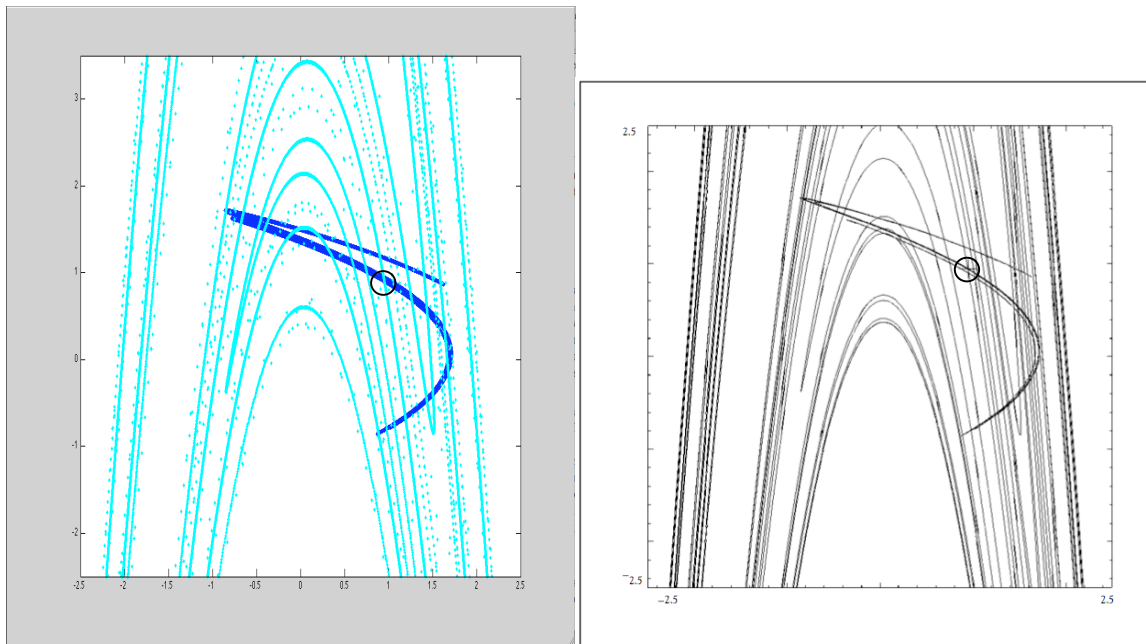
At this point, we take a step back and ask what we would like to say about the homoclinic tangle. It certainly *seems* intuitively chaotic, with the stable and unstable manifolds crossing transversely an infinite number of times. However, we don't yet have the formalism that allows us to make this claim. In order to substantiate this claim, and to make other claims about points *near* a homoclinic orbit, we introduce the idea of the Smale Horseshoe Map.

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[1] referenced in *Chaos*, Chapter 10.

[2] *Chaos*, Figure 10.7.

III. Smale Horseshoe Map

Stephen Smale (born 1930) is an American mathematician who has published numerous works on dynamical systems. In 1961, he proved the Poincaré Conjecture for all dimensions greater than four.[3] In 1967, published "Differentiable Dynamical Systems,"[4] a study that included his famous *Smale Horseshoe* model.

This map is essentially a generalized, continuous version of the Skinny Baker map. This difference, however, is key because its continuity allows us to make several formal claims about the map. In the diagram below,[5] it maps the square Z=ABCD to the overlapping horseshoe $A^*B^*C^*D^*$. Inside the square, we make the assumption that it "uniformly contracts distances horizontally and expands distances vertically"[6]. This assumption will later allow us to make formal claims about its Lyapunov exponents.
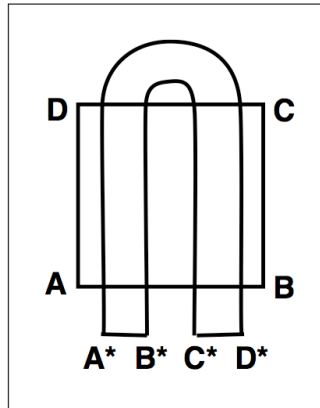
Now, in order to visualize the map of f^k, we ask what the maps f^1 and f^2 do to the same square Z.

Since f maps the square Z onto the horseshoe, then f^{-1} maps the horseshoe onto Z. Informally, we can "deform" or "reshape" $A^*B^*C^*D^*$ from a horseshoe shape into a square (to determine what f^{-1} does to the *square* $A^*B^*C^*D^*$). To do so, we must also deform its image ABCD in the same way, since both f and f^{-1} are continuous functions.
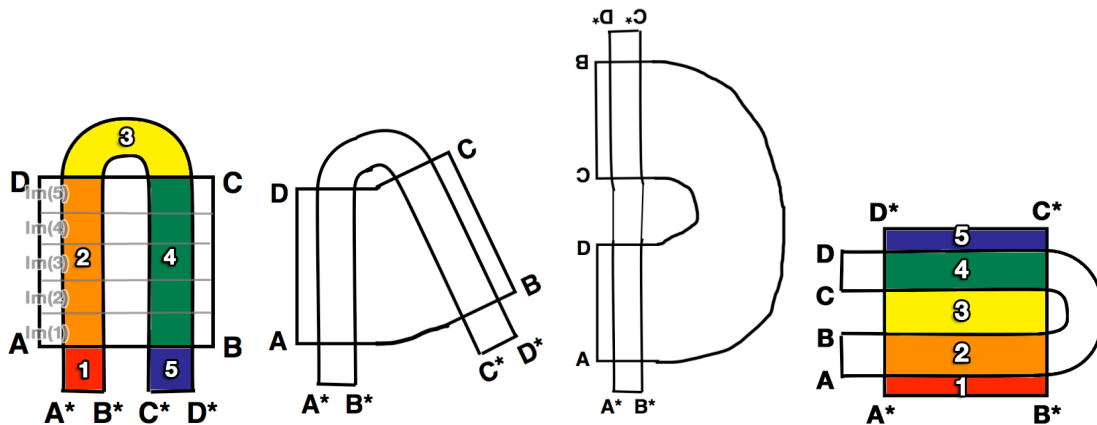
[3] Smale, S. *Generalized Poincaré's conjecture in dimensions greater than four.*

[4] Smale, S. *Differentiable Dynamical Systems.*

[5] *Chaos,* Figure 5.15.

[6] *Chaos,* Chapter 5, 216.

With slightly more formalism, we can divide $A^*B^*C^*D^*$ up into five bands (labeled 1 through 5 in the diagram above). The points in bands 1, 3, and 5 are not contained in the image of $A^*B^*C^*D^*$ under f^1. By contrast, the points in bands 2 and 4 *are* contained in the image of $A^*B^*C^*D^*$. These bands (which can be seen as horizontal bands in the square) are the only part of the image that concerns us when we make our claims later, so this formalization is sufficient.

The map of f^2 is clear under inspection. It forms the image below[7]. If we consider the segments of $f^k(Z)$ that overlap Z, we notice that each subsequent iteration of f will double the number of overlapping vertical lines, and shrink their widths.
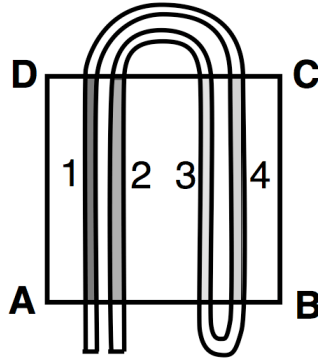
If we note that $(f^k(Z) \cap Z)$ contains $(f^j(Z) \cap Z)$ for all $0<k<j$, and that $(f^{-k}(Z) \cap Z)$ contains $(f^{-j}(Z) \cap Z)$ for all $0<k<j$, we can make the following claim:

The overlapping region of Z and $f^k(Z)$, for any integer k, represents the region that remains in Z under k iterations of f.

Then the intersection $(f^k(Z) \cap Z \cap f^k(Z))$ represents the regions of Z that remain inside Z under k iterations of f or f^1. So if we take the limit $\lim_{k \to \infty}(f^k(Z) \cap Z \cap f^k(Z))$, this will give us the pieces of Z that remain inside Z under infinite iterations of f or f^1. These are called **nonwandering points**. The intersection $f^2(Z) \cap Z \cap f^{-2}(Z)$ is plotted below.[8]

D

D ∧ F(D)

D ∧ F(D) ∧ F²(D)

D ∧ F¹(D)

D ∧ F¹(D) ∧ F²(D)

F²(D) ∧ F¹(D) ∧ D ∧ F(D) ∧ F²(D)
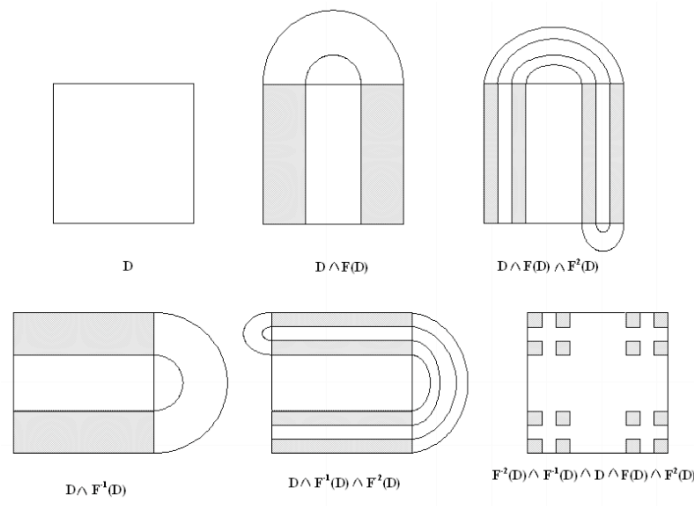
[7] *Chaos*, Figure 5.17(a)

[8] "Chaos in Continuous Dynamical Systems." Lecture Notes.

The limit of this intersection as k→∞ is a version of **Cantor dust**: the Cantor set crossed with itself.[9]
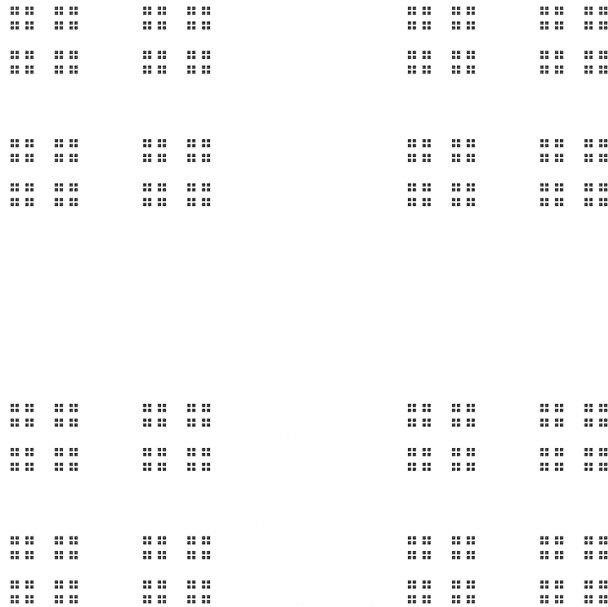
With this formalism of non-wandering points, we can make several claims about the horseshoe map[10]:

i.   It contains a countable set of periodic orbits of arbitrarily long periods.
ii. It contains an uncountable set of bounded non periodic orbits.
iii. It contains a chaotic orbit.

These claims come from analogues of properties we proved about the Skinny Baker map. To justify the third claim, we note that our "uniform expansion/contraction" assumption inside the square guarantees nonzero Lyapunov exponents, with one Lyapunov exponent greater than 0. This implies chaos for all orbits that aren't asymptotically periodic. Such orbits exist, and correspond to irrational initial conditions in the Skinny Baker map.

IV. Formulation of Horseshoe Map from Homoclinic Orbit

Now we will formulate the horseshoe map from a homoclinic orbit.

Pick some nonnegative distance d.
Since p' is on the stable manifold S, it will map (along S) within d of p after some number of iterations of f. Call this number x.
Since p' is on the unstable manifold U, it will map (along U) within d of p after some number of iterations of f^{-1}. Call this number y.

[9] Image from http://en.wikipedia.org/wiki/Cantor_set
[10] "Chaos in Continuous Dynamical Systems." Lecture Notes.

Consider a box Z around the origin that extends a distance *d* (small) along each manifold. If we iterate f on this box x times, p will remain in the box, but the box will stretch along the stable manifold to cover p'. Thus, p'∈f^x(Z). If we iterate f^1 on this box y times, p will remain in the box, but the box will stretch along the unstable manifold to cover p'. Thus, p'∈f^{-y}(Z).


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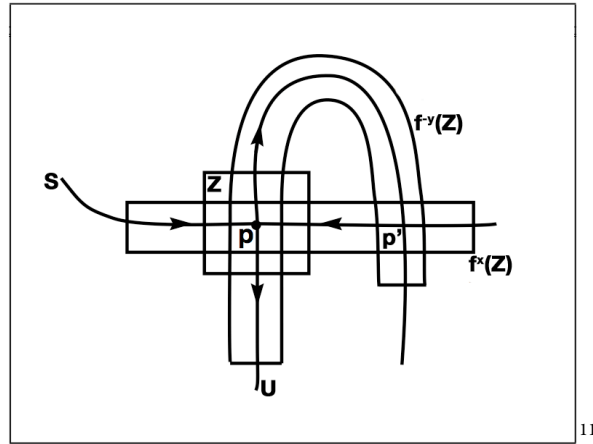
Then consider this box f^{-y}(Z). If we call the function f^{x+y} on this box, we will end up with the box f^x(Z). Since they must cross over each other as in the diagram above, we have just found a map that is equivalent to the general horseshoe map, under a change in coordinates.

I implemented this technique of "box iteration" in MATLAB. The code (*project4.m*) can be found at the end of the paper, and its application to the Hénon map from Section II can be seen below. All boxes are drawn in red. In this case, for a *d* value of 0.5, I found that the first valid *x*=2 and the first valid *y*=4. As a result of this number of iterations and the different stretching factors, the boxes along the unstable manifold are stretched to a much smaller width than those along the stable manifold. Nevertheless, the overlapping box can be seen in the second figure, enlarged from the black box in the first figure. The two closely-grouped horizontal lines intersect the vertical strip, and the intersection includes a homoclinic point.

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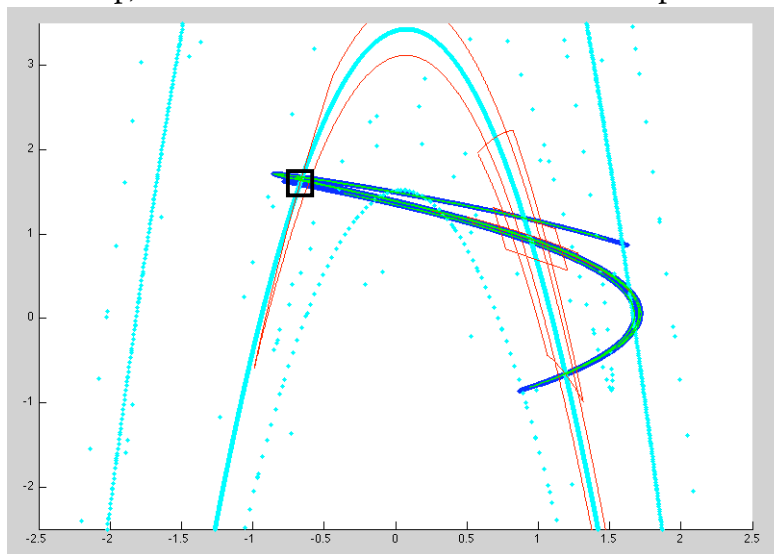
[11] Image modified from *Chaos,* Figure 10.11.
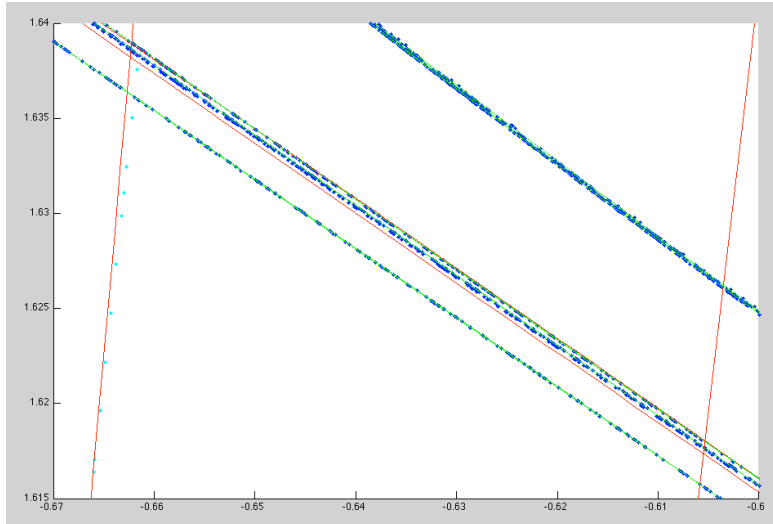
The green lines represent the next iteration of f^{x+y} on the box $f^x(Z)$, so all non-wandering points must be inside one of the green regions, as well. (This region is smaller than the horizontal red region, and is almost just a single pixel's width in this view.)

V. Conclusion

Recall now our previous results about the horseshoe map:
  i.   It contains a countable set of periodic orbits of arbitrarily long periods.
  ii.   It contains an uncountable set of bounded non periodic orbits.
  iii. It contains a chaotic orbit.

As a result of the comparison between the horseshoe map and the homoclinic tangle, we can now state that a single transverse homoclinic point implies the existence of (i), (ii), and (iii). We can't predict the exact locations of these orbits directly from this result, but their existence is guaranteed. Notably, we have now proven our initial goal: a single transverse homoclinic point implies chaotic orbits.

Code from project.m:

```matlab
% Matt Harding
% Math 53 Project:
% The Homoclinic Tangle and Smale Horseshoe Map
% project.m : Approximation of Stable, Unstable Manifolds

f=@(x) [2.12 - x(1).^2 - 0.3*x(2); x(1)]; % Henon map, Example 10.5
finv=@(x) [x(2); (-10/3)*(x(2).^2 + x(1) - 2.12)]; % Inverse

p = [0.944521871910198; 0.944521871910198]; % fixed point p

u_vec = [1; -0.58]; % unstable eigenvector
s_vec = [1; -5.71]; % stable eigenvector

% Normalize eigenvectors
u_vec = u_vec./sqrt(sum(u_vec.^2));
s_vec = s_vec./sqrt(sum(s_vec.^2));


% PART 1: UNSTABLE
x = 400; % number of points along each segment (e.g., along J = ab)
n = 400; % number of iterations of f

a_u = zeros(2, x, n); % 2 dimensions, x points, n iterations
a_u(:,1,1)= p + u_vec.*(1e-6); % initial point along unstable eigvec

for d = 2:n % iterate f n times to get each starting/ending point
    a_u(:,1,d) = f(a_u(:,1,d-1));
end

dir_u = a_u(:,1,2)-a_u(:,1,1); % direction of J
for i = 2:x
    a_u(:,i,1) = a_u(:,i-1,1) + (1/x)*(dir_u); % plot all x pts along ab
    for d = 2:n
        a_u(:,i,d) = f(a_u(:,i,d-1)); % plot all pts along subsequent iters
    end
end

figure;
hold on;
for d = 1:n
    plot(a_u(1,:,d), a_u(2,:,d), '.b');
end

% PART 2: STABLE
x = 1e6; % number of points along each segment (e.g., along J = ab)
n = 50;  % number of iterations of f

a_s = zeros(2, x, n); % 2 dimensions, x points, n iterations
a_s(:,1,1)= p + s_vec.*(1e-7); % initial point along stable eigvec

for d = 2:n % iterate f-1 n times to get each starting/ending point
    a_s(:,1,d) = finv(a_s(:,1,d-1));
end

dir_s = a_s(:,1,2)-a_s(:,1,1); % direction of J
for i = 2:x
    a_s(:,i,1) = a_s(:,i-1,1) + (1/x)*(dir_s); % plot all x pts along ab
    for d = 2:n
        a_s(:,i,d)=finv(a_s(:,i,d-1)); % plot all pts along subsequent iters
    end
end

hold on;
for d = 1:n
    plot(a_s(1,:,d), a_s(2,:,d), '.c');
end

axis([-2.5 2.5 -2.5 3.5]);
```

Added code from project4.m:

```matlab
% BOX AROUND FIXED POINT
num = 1000; % pts on a side
len = 0.5; % side length
sideA = zeros(2,num);
sideB = zeros(2,num);
sideC = zeros(2,num);
sideD = zeros(2,num);

sideAinv = zeros(2,num);
sideBinv = zeros(2,num);
sideCinv = zeros(2,num);
sideDinv = zeros(2,num);

sideA(:,1) = p + u_vec.*(-len/2) + s_vec.*(len/2);
sideB(:,1) = p + u_vec.*(len/2) + s_vec.*(len/2);
sideC(:,1) = p + u_vec.*(-len/2) + s_vec.*(-len/2);
sideD(:,1) = p + u_vec.*(-len/2) + s_vec.*(len/2);

% draw box
for i = 2:num
    sideA(:,i) = sideA(:,i-1) + u_vec.*(len/num);
    sideB(:,i) = sideB(:,i-1) + s_vec.*(-len/num);
    sideC(:,i) = sideC(:,i-1) + u_vec.*(len/num);
    sideD(:,i) = sideD(:,i-1) + s_vec.*(-len/num);
end

plot(sideA(1,:), sideA(2,:), '-r');
plot(sideB(1,:), sideB(2,:), '-r');
plot(sideC(1,:), sideC(2,:), '-r');
plot(sideD(1,:), sideD(2,:), '-r');

numiter = 4;    % number of forward iterations on the box
numiterinv = 2; % number of backward iterations on the box

%set up "inverse" box
sideAinv(:,:) = sideA(:,:);
sideBinv(:,:) = sideB(:,:);
sideCinv(:,:) = sideC(:,:);
sideDinv(:,:) = sideD(:,:);

for j = 1:numiter % plot f(box)
    for i = 1:num
        sideA(:,i) = f(sideA(:,i));
        sideB(:,i) = f(sideB(:,i));
        sideC(:,i) = f(sideC(:,i));
        sideD(:,i) = f(sideD(:,i));
    end
    plot(sideA(1,:), sideA(2,:), '-r');
    plot(sideB(1,:), sideB(2,:), '-r');
    plot(sideC(1,:), sideC(2,:), '-r');
    plot(sideD(1,:), sideD(2,:), '-r');
end

for j = 1:numiterinv % plot f-1(box)
    for i = 1:num
        sideAinv(:,i) = finv(sideAinv(:,i));
        sideBinv(:,i) = finv(sideBinv(:,i));
        sideCinv(:,i) = finv(sideCinv(:,i));
        sideDinv(:,i) = finv(sideDinv(:,i));
    end
    plot(sideAinv(1,:), sideAinv(2,:), '-r');
    plot(sideBinv(1,:), sideBinv(2,:), '-r');
    plot(sideCinv(1,:), sideCinv(2,:), '-r');
    plot(sideDinv(1,:), sideDinv(2,:), '-r');
end

% Find a smaller region where nonwandering points reside
% by iterating f or f-1 another (x+y) times
for j = 1:(numiter+numiterinv)
    for i = 1:num
        sideA(:,i) = f(sideA(:,i));
        sideB(:,i) = f(sideB(:,i));
        sideC(:,i) = f(sideC(:,i));
        sideD(:,i) = f(sideD(:,i));
    end
end

plot(sideA(1,:), sideA(2,:), '-g');
plot(sideB(1,:), sideB(2,:), '-g');
plot(sideC(1,:), sideC(2,:), '-g');
plot(sideD(1,:), sideD(2,:), '-g');
```

References

[1]   Alligood, Sauer, and Yorke. *Chaos: An Introduction to Dynamical Systems.* First edition. Springer, 1996.

[2]   S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Annals of Mathematics, 2nd Series, 74 (1961), no. 2, 391 – 406, referenced by http://en.wikipedia.org/wiki/Stephen_Smale.

[3]   Smale, S. *Differentiable Dynamical Systems*. Bulletin of the American Mathematical Society. Volume 73, Number 6 (1967), 747-817.
http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.bams/1183529092

[4]   University of Wisconsin. "Chaos in Continuous Dynamical Systems." Lecture notes.
http://www.uwosh.edu/faculty_staff/kuennene/Chaos/ChaosNotes11.pdf

[5]   "Cantor Set." Wikipedia. http://en.wikipedia.org/wiki/Cantor_set

[6]   You, Kostelich, and Yorke. "Calculating stable and unstable manifolds." 1991. International Journal of Bifurcation and Chaos.