Numerical Analysis of the Dynamics of a Double Pendulum

Roja Nunna

Math 53: Chaos!

Professor Alexander Barnett

Dartmouth College

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A planar double pendulum is a simple mechanical system that has two simple pendula attached end to end that exhibits chaotic behavior. The aim of this project will be to numerically analyze the dynamics of the double pendulum system. First, the physical system is introduced and a system of coordinates is fixed, and then the Lagrangian and the Hamiltonian equations of motions are derived. We will find that the system is governed by a set of coupled non-linear ordinary differential equations and using these, the system can be simulated. Finally we analyze Poincaré sections, the largest lyapunov exponent, progression of trajectories, and change of angular velocities with time for certain system parameters at varying initial conditions.

System of coordinates

The double pendulum is illustrated in Fig. 1. It is convenient to define the coordinates in terms of the angles between each rod and the vertical. In this diagram m_1, L_1, and θ_1 represent the mass, length and the angle from the normal of the inner bob and m_2, L_2, and θ_2 stand for the mass, length, and the angle from the normal of the outer bob. To simply our numerical analysis, let us assume that $m_1 = m_2 = m$ and $L_1 = L_2 = l$. That is, we consider two identical rods with ($I = \frac{1}{12}ml^2$). Assume that masses of rods can be neglected but their moment of inertia should be included to better reflect the physical system they represent[1].

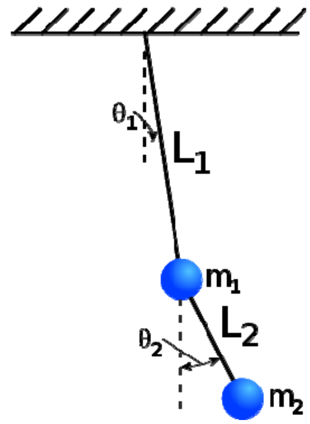

Figure 1: Schematic representation of the construction of a simple double pendulum where m_1, L_1, and θ_1 represent the mass, length and the angle from the normal of the inner bob and m_2, L_2, and θ_2 stand for the mass, length, and the angle from the normal of the outer bob[2].

Equations of motion

Now, by resolving these quantities onto horizontal and vertical components, we obtain the position of the center of mass of the two rods, where(x_1, y_1) are the position of the inner bob and (x_2, y_2) is the position of the outer bob.

$$x_1 = \frac{l}{2}\sin(\theta_1) \rightarrow (1)$$

$$x_2 = l\left(\sin(\theta_1) + \frac{1}{2}\sin(\theta_2)\right) \rightarrow (2)$$

$$y_1 = -\frac{l}{2}\cos(\theta_1) \rightarrow (3)$$

$$y_2 = -l\left(\cos(\theta_1) + \frac{1}{2}\cos(\theta_2)\right) \rightarrow (4)$$

The Lagranian is given by $L = Kinetic\ Energy - Potential\ Energy$

$$L = \frac{1}{2}m\,(v_1^2 + v_2^2) + \frac{1}{2}I\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - mg(y_1 + y_2) \rightarrow (5)$$

$$L = \frac{1}{2}m\,(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - mg(y_1 + y_2) \rightarrow (6)$$

The first term is the linear kinetic energy of the center of mass of the bodies and the second term is the rotational kinetic energy around the center of mass of each rod. The last term is the potential energy of the bodies in a uniform gravitational field.

Plugging in the coordinates above, we obtain

$$L = \frac{1}{6}ml^2\left[\dot{\theta}_2^2 + 4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + \frac{1}{2}mgl\,(3cos\theta_1 + cos\theta_2) \rightarrow (7)$$

There is only one conserved quantity (the energy), and no conserved momenta. The two momenta may be written as

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = \frac{1}{6}ml^2\left[8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1-\theta_2)\right] \rightarrow (8)$$

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = \frac{1}{6}ml^2\left[2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1-\theta_2)\right] \rightarrow (9)$$

These expressions may be inverted to get

$$\dot{\theta}_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3cos\,(\theta_1-\theta_2)p_{\theta_2}}{16-9cos^2(\theta_1-\theta_2)} \rightarrow (10)$$

$$\dot{\theta}_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3cos\,(\theta_1-\theta_2)p_{\theta_1}}{16-9cos^2(\theta_1-\theta_2)} \rightarrow (11)$$

The remaining equations of motion for momentum are

$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial\theta_1} = -\frac{1}{2}ml^2\left[\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + 3\frac{g}{l}sin\theta_1\right] \rightarrow (12)$$

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial\theta_2} = -\frac{1}{2}ml^2\left[-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \frac{g}{l}sin\theta_2\right] \rightarrow (13)$$ [3]

Let's assume now that $m = l = 1$. This gives us a set of four equations that can be used to simulate the behavior of the double pendulum

$$\dot{\theta}_1 = 6\frac{2p_{\theta_1} - 3cos\,(\theta_1-\theta_2)p_{\theta_2}}{16-9cos^2(\theta_1-\theta_2)} \rightarrow (14)$$

$$\dot{\theta}_2 = 6\frac{8p_{\theta_2} - 3cos\,(\theta_1-\theta_2)p_{\theta_1}}{16-9cos^2(\theta_1-\theta_2)} \rightarrow (15)$$

$$\dot{p}_{\theta_1} = -\frac{1}{2}\left[\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + 3gsin\theta_1\right] \rightarrow (16)$$

$$\dot{p}_{\theta_2} = -\frac{1}{2}\left[-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + gsin\theta_2\right] \rightarrow (17)$$

The conserved quantity, energy function, is given by $Hamiltonian = Kinetic\ Energy + Potential\ Energy$

$$H = \theta_i p_i - L = ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) - 2mglcos\theta_1 - mglcos\theta_2 \rightarrow (18)$$

Substituiting the above solved equations for $\dot{\theta}_1$ and $\dot{\theta}_2$, we obtain

$$H = \frac{3ml^2 {p_{\theta_2}}^2 - 2ml^2 p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2ml^4[m + m\,sin^2(\theta_1-\theta_2)]} - 2mglcos\theta_1 - mglcos\theta_2 \rightarrow (19)$$

Again, when m=l=1,

$$H = \frac{3{p_{\theta_2}}^2 - 2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2[1+sin^2(\theta_1-\theta_2)]} - 2gcos\theta_1 - gcos\theta_2 \rightarrow (20)$$ [4]

Simulation of the motion for various initial conditions can be performed by running the attached MATLAB code that uses ode45.

Some Theory

Let us introduce a new concept of quasi periodic motion. Roughly speaking, they refer to "almost periodic" motion. More mathematically, it can be thought of as [5]the type of motion executed by a dynamical system containing a finite number of incommensurable frequencies.

From equations 14 to 17, we see that the dynamics of a double pendulum can be described with 4 variables, the two angles and their corresponding (angular) velocities, which span the four-dimensional phase space of the system[6]. Since the double pendulum is a hamiltonian system, total energy is conserved (equation 20) and this reduces the four-dimensional phase space to a three-dimensional manifold. Further, The KAM theorem states that if a hamiltonian system is subjected to a weak nonlinear perturbation, some of the invariant tori that have "sufficiently irrational" frequencies survive. In other words, the motion continues to be quasiperiodic[7]. KAM tells us that at lower energies, the function is integrable (it has as many conserved quantities as there are degrees of freedom in the system)[8]. At high energy the pendulum behaves like a simple rotor, with the system rotating rapidly in a stretched cofiguration ($\theta_1 = \pi, \theta_2 = 0$). In this case the kinetic energy terms in the Lagrangian dominate the potential energy terms and may be described by setting g = 0 in the equations of motion. The total angular momentum is conserved, because in the absence of gravity, there is no torque on the pendulum. The resulting motion of the system is regular (non-chaotic), because a system with two degrees of freedom and two constraints (conservation of total energy and total angular momentum) cannot exhibit chaos. It follows, for example, that the double square pendulum would not exhibit chaos if installed on the space station[9]. Lower energies and higher energies = periodic motion. From this theoretical evidence, we hypothesize that the behavior of a double pendulum varies from regular motion at low energies, to chaos at intermediate energies, and back to regular motion at high energies[10].

Poincare

Poincare sections allow fast and informative insight into the dynamics of the double pendulum. The different types of motion appear as finite number of points for periodic orbits, curve filling points ('invariant curves') for quasi periodic motion and area filling points for chaotic trajectories.

We can construct a two-dimensional Poincare section by looking at the trajectory only at those points when the outer pendulum passes the vertical position, that is $\theta_2 = 0$. Equation 20 then yield s a quadratic equation for θ_2, with solutions

$$\dot{\theta}_{2\pm} = -\dot{\theta}_1 cos\theta_1 \pm \sqrt{\left(\dot{\theta}_1 cos\theta_1\right)^2 - \frac{2}{ml^2}\left(mgl(1 - cos\theta_1) + \frac{1}{2}ml^2\dot{\theta}_1{}^2 - H\right)}$$

→(21)

We can now plot a $(\theta_1, \dot{\theta}_1)$ in the phase space of the inner pendulum when the two conditions $\dot{\theta}_2 = 0$ and $\dot{\theta}_1 cos\theta_1 > 0$ are fulfilled.

Largest Lyapunov Exponent

Sensitive dependence on initial conditions – small separations between arbitrarily close initial conditions are amplified exponentially in time – is the hallmark of chaos. The underlying cause of this behavior, namely the exponential growth, can be numerically and analytically evaluated using lyapunov exponents. Largest lyapunov exponents[11], as it effectively gives us the information on the divergence of two close trajectories. We can use the same first order equations used in the MATLAB simulation to evaluate the exponent. The method to calculate the lyapunov exponent is to first plot the natural logarithm of the separation between the two closely launched trajectories against time and then find the slope of the region where it is increasing. As usual, positive lyapunov exponents are indicative of chaotic behavior.

Other informative plots

We can also plot the four variables that characterize the system against each other to get a qualitative sense of the behavior of the system. It is more difficult to gauge what is happening with the dynamics of the system with such plots but they are still indicative of periodic versus chaotic behavior.

Now we have all the tools to look at some simulations and see if the theory fits the observed behavior.

Simulations

Let us start with low energy conditions. Using the Hamiltonian, the energy can easily be calculated in Joules. When we have initial conditions = y0 = $\left[\dot{\theta}_1, \dot{\theta}_2, \dot{p}_{\theta_1}, \dot{p}_{\theta_2}\right]$ = [0.2,0.2828,0,0], the energy = 0.7809 J. At this low energy we expected periodic behavior. The attached code titled 'hamiltonian.m' returns this value for an input of y0.

(Figures 2, 3, 4, 5, 6 are in Appendix A) The periodic trajectory of the outer bob is clear from figure 2a, the Poincare sections are presented in figure 2b and show a finite number of points that grow outwards with time but form a general pear shape. The plot in 3-D forms a pear shape when rotated about the x_1 axis. When a second trajectory is launched at a distance of 10^{-9} from this initial condition, we see from figure 2d that they move together indicating that there is no chaos. When the lyapunov graph is plotted, it is clear that the lyapunov exponent is negative (= -3.426) and hence the system is not chaotic. The angular velocities of the inner and outer bobs are in phase and periodic further confirming that these initial conditions at low energy are indicative of the non-chaotic regime.

As we increase the energy to 1.2807J with the intial conditions y0 = [0.7,0.3825,0,0], we enter the quasi periodic regime. The periodic trajectory of the outer bob is in figure 3a and we see that the inner and the outer bob are out of phase (also exemplified in the angular velocities graph in 3d. Figure 3e shows that two closely launched trajectories do not diverge from each other. Using this distance separation, when the lyapunov graph was generated in 3e, we see that the lyapunov exponent does not remain entirely negative and starts to begin increasing to positive values with time. The average lyapunov exponent is -1.203 which is higher than for the periodic condition.

When the energy is further increased to be 29.4 J with initial conditions y0 = [π,π,0.5,0]. We repeat the same process and find a average positive lyapunov exponent of 1.906 (figure 4e). This average number was obtained by calculating the average over a range of initial conditions around this y0 (figure 4f). We see in figure 4a that the path of the outer bob is random and unpredictable. The Poincare section is clearly area filling and the two closely launched trajectories (with an initial separation of 10^{-9}) start off together but move away from each other. The angular velocities which are qualitative indicators, also show that there is no regular behavior. All of these clearly indicate that at these energies, the system is chaotic.

So far we found that as we increase energy, the system moves from being periodic to quasi-periodic to chaos. Now, if we increase the energy even more to reach 104.25 J with initial conditions y0 = [π, 0, 0.5, 0.5], we see the quasi-periodic state again! Clearly, from 5d the trajectories have not diverged from each other. Also, there is a decrease in the lyapunov exponent to 0.4320. The angular velocities (figure 5c) are out of phase together. The trajectory in figure 5a is a certain indication of quasi-periodic motion. Although the Poincare sections are a little difficult to understand, the other graphs are indicative of quasi-periodic behavior at these energies.

Let us take a look at figure 6, that has other informative plots of the variables plotted against each other. There is qualitative trend indicating the system starts out to be periodic, in the sense that the shape of the curves can be predicted for later times, to chaotic unpredictable behavior and finally back to shapes resembling periodicity. In each of them, the top left is the angles of the inner and outer bob, top right has the angular velocities of the inner and the outer bob. Bottom and left and right show the angle versus angular velocity of the inner bob and the outer bob respectively.

The graphs were generated using the codes PoincareTrajecAngleEnergy.m and crudelyap.m. They need Hamiltonian.m, doublependulum2.m and zerocross2.m to work. All the code is attached in Appendix B.

Appendix A: Figures

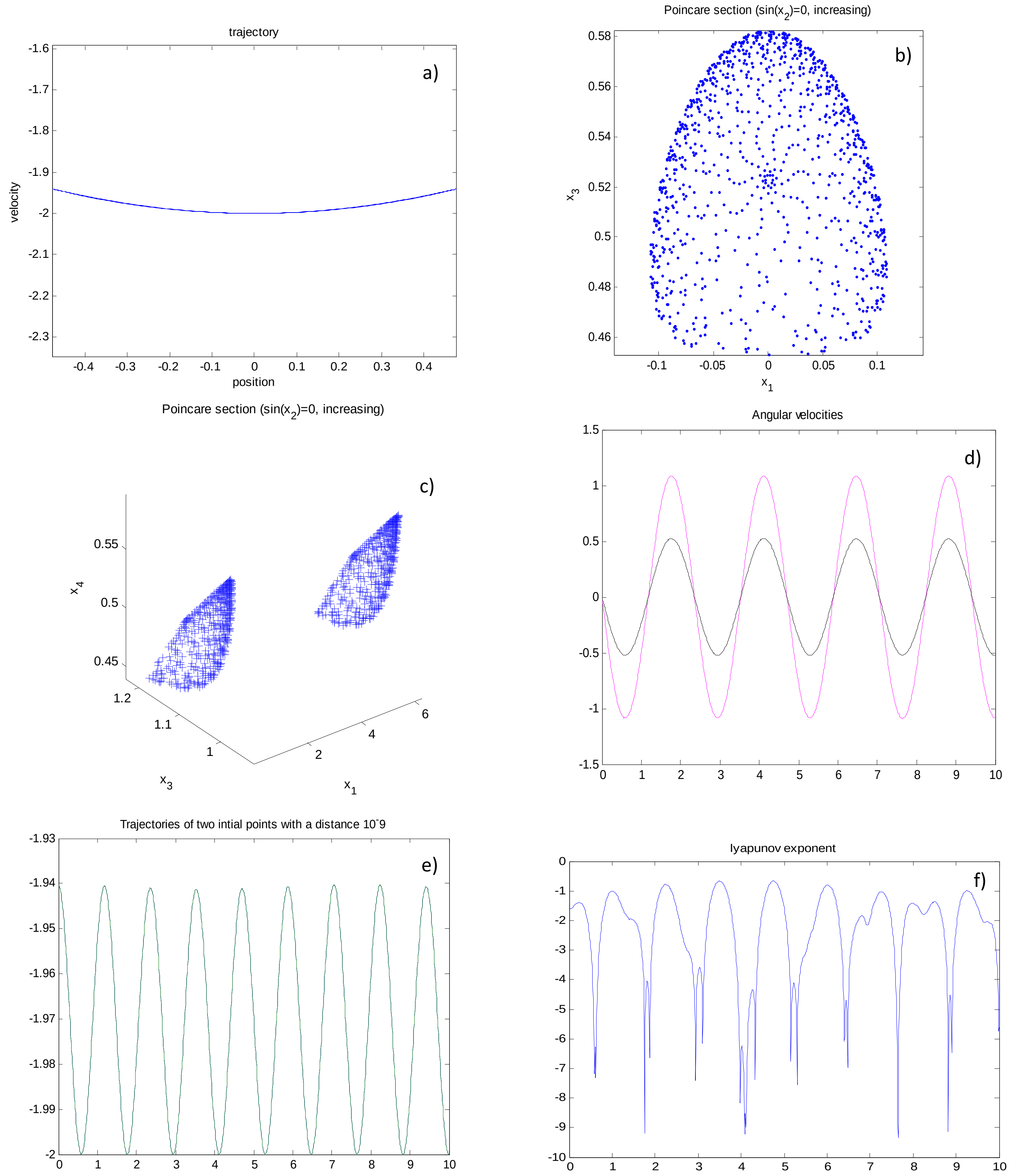

Figure 2: y0 = [0.2,0.2828,0,0] produces periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $\theta_2 = 0$; c) 3-D Poincare map when the position of the outer bob is at zero i.e $\sin\theta_2=0$; d) angular velocities of the outer and the inner bob against time; e) Trajectories of two curves at a distance of 10^{-9} between them; f) Illustrates the negative lyapunov exponent.

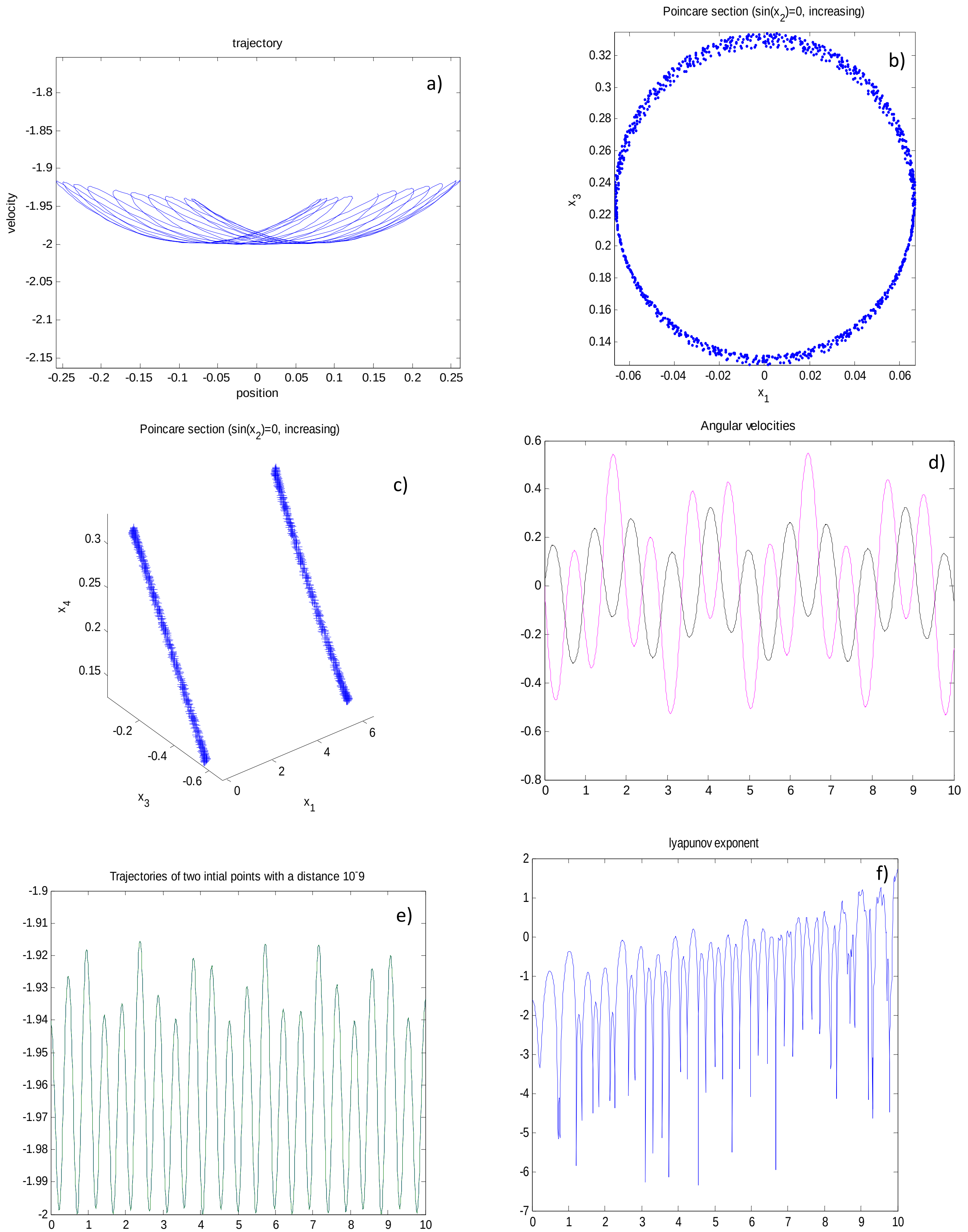

Figure 3: y0 = [0.2,- 0.2828,0,0] produces quasi-periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $\theta_2 = 0$; c) 3-D Poincare map when the position of the outer bob is at zero i.e $\sin\theta_2=0$; d) angular velocities of the outer and the inner bob against time; e) Trajectories of two curves at a distance of 10^{-9} between them; f) slowly increasing lyapunov exponent.

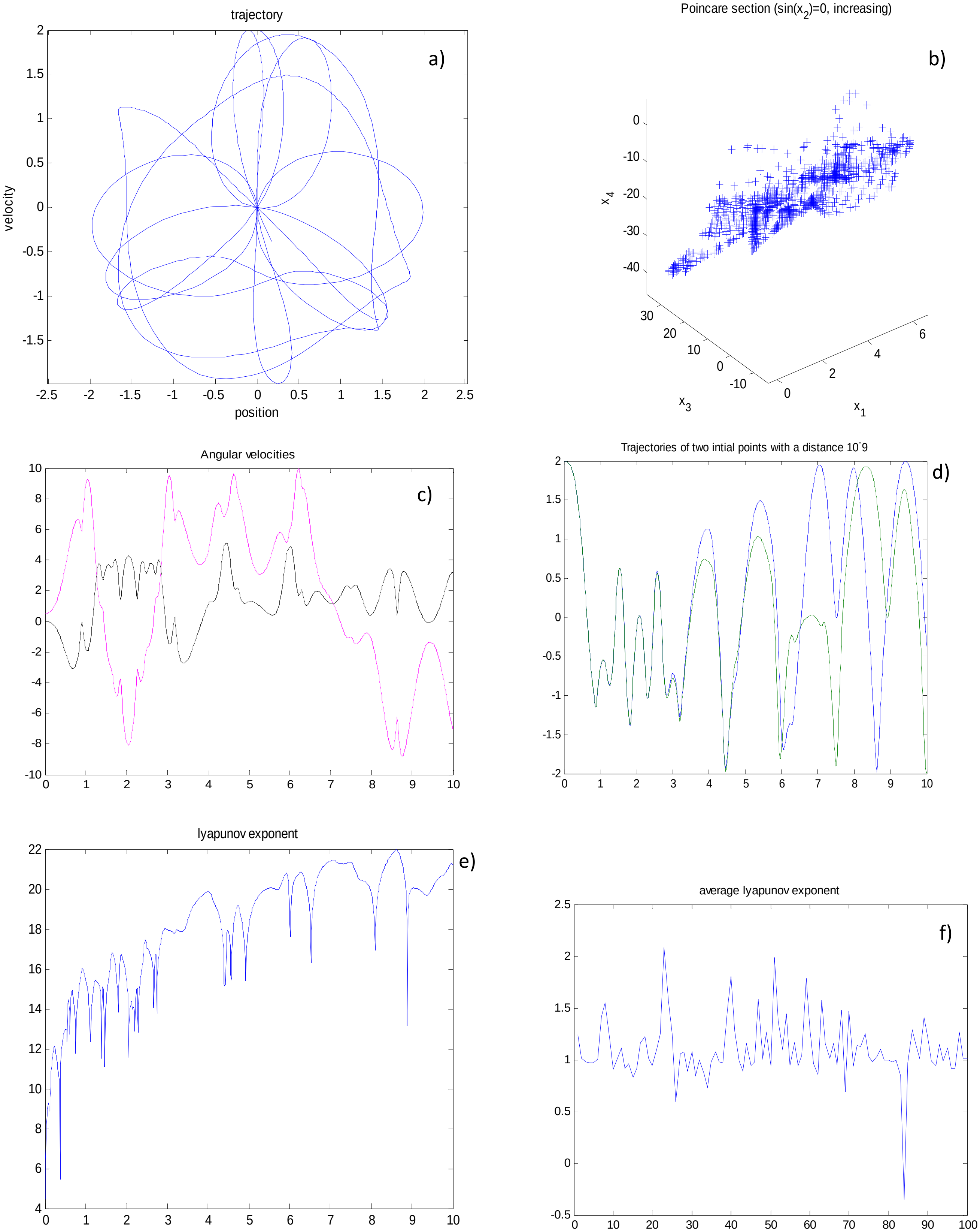

Figure 4: y0 = [π,π,0.5,0] produces chaos! a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $\theta_2 = 0$; c) angular velocities of the outer and the inner bob against time; d) Trajectories of two curves at a distance of 10^{-9} between them; e) positive lyapunov exponent; f) average lyapunov exponent for various initial conditions.

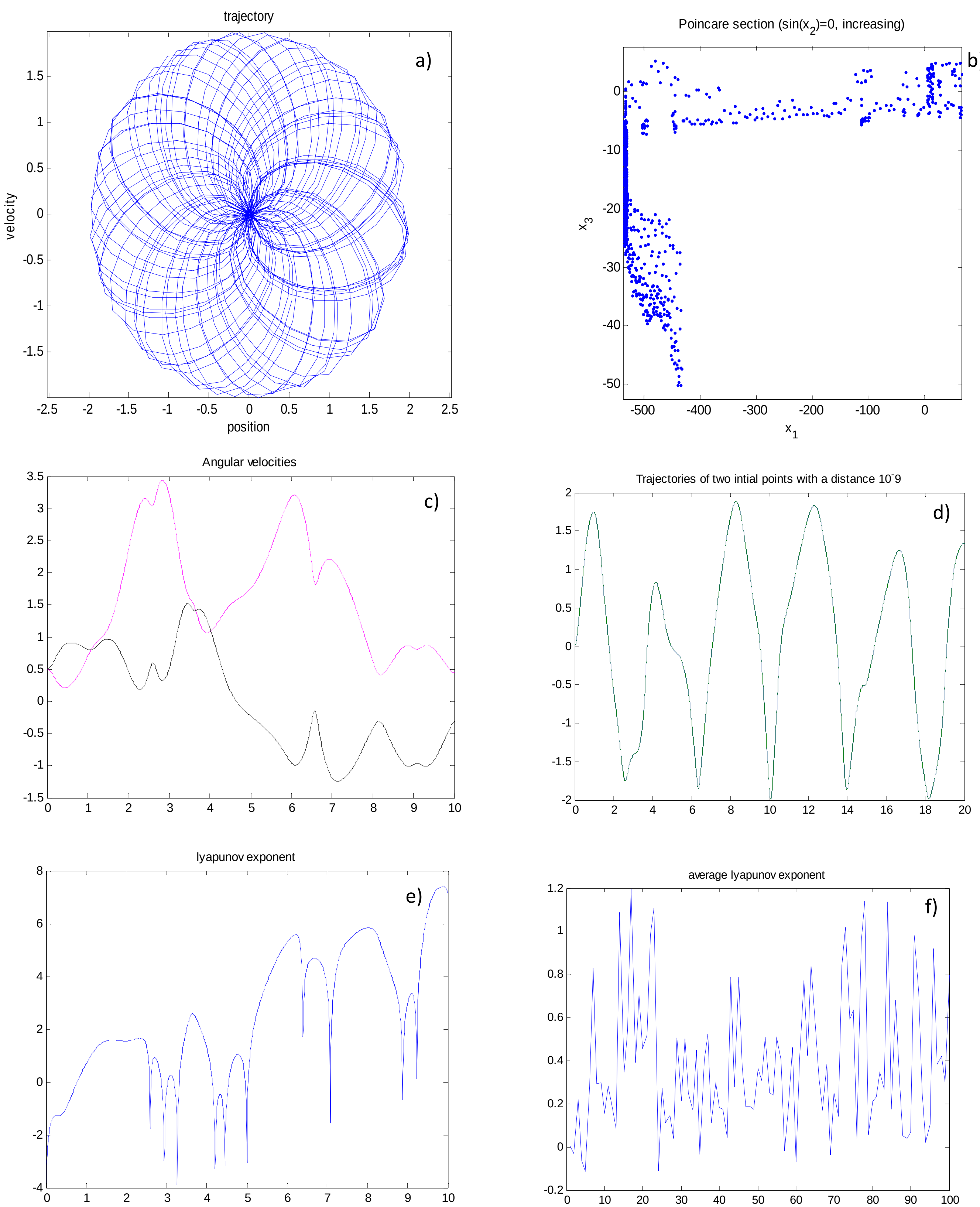

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Figure 5: y0 = [π,0,0.5,0.5] produces quasi-periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $\theta_2 = 0$; c) angular velocities of the outer and the inner bob against time; d) Trajectories of two curves at a distance of 10^{-9} between them; e) slightly positive lyapunov exponent; f) average lyapunov exponent for various initial conditions.

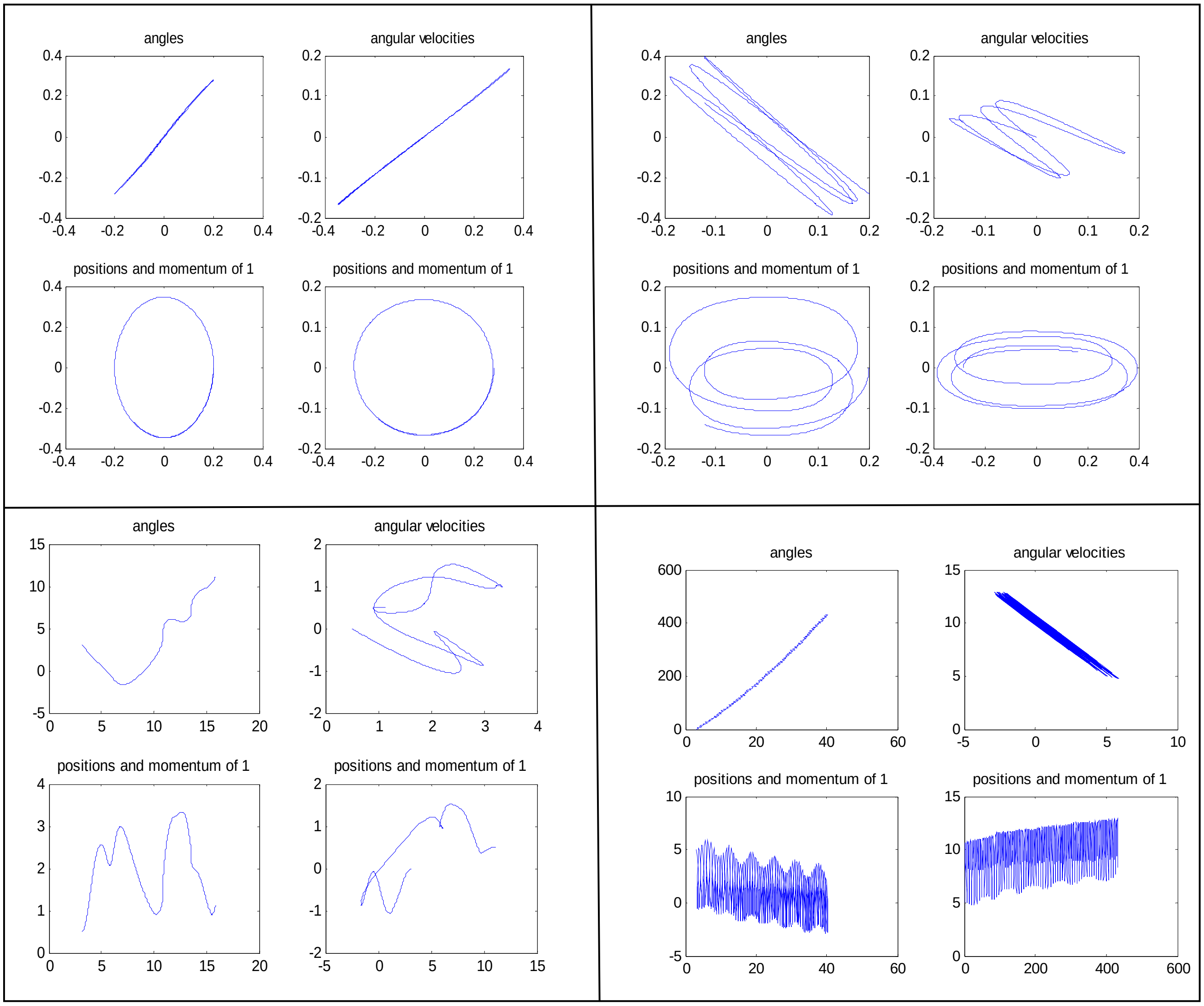

Figure 6: In each square, the top left is the angles of the inner and outer bob, top right has the angular velocities of the inner and the outer bob. Bottom and left and right show the angle versus angular velocity of the inner bob and the outer bob respectively. The top left square shows periodic behavior, the top right square indicates quasi-periodic aka almost periodic behavior, the bottom left square has the chaotic system depicted and the bottom right square starts to having resemblances to periodic behavior.

Appendix B: MATLAB codes

Animation of the pendulum

doublependulumAnimation.m

```
% modified from http://math.la.asu.edu/~kawski/MATLAB/matlab.html
%comments by Roja

function out=dpend()
if nargin < 5
   n=3;
end
%declare variables
global m1 m2 L1 L2 g;
m1 = 2;
m2 = 1;
g  = 1;
L1 = 1;
L2 = sqrt(3);
dt = 0.1;
y0 = [-pi 0 -pi 2*rand(1,1)-1];
t0 = 0;
twin=30;
r1 = (L1+L2)/20*sqrt(m1/(m1+m2));
r2 = (L1+L2)/20*sqrt(m2/(m1+m2));
%size the animation window
LL = 1.2 * ( L1 + L2 );
% re-use data while disk is being plotted
bx = cos(pi*[0:0.1:2]);
by = sin(pi*[0:0.1:2]);

% initial positions (delete previous images later)
x1= L1*sin(y0(1));
y1=-L1*cos(y0(1));
x2=x1+L2*sin(y0(3));
y2=y1-L2*cos(y0(3));
b4=plot([0,x1],[0,y1],'k-','LineWidth',2);
hold on
b3=plot([x1,x2],[y1,y2],'k-','LineWidth',2);
b1=fill(x1+r1*bx,y1+r1*by,'magenta');
b2=fill(x2+r2*bx,y2+r2*by,'black');
% scale the window to fit the double pendulum
axis([-LL LL -LL LL])

for N=[1:1000]
     % save the old angles and positions
     posold=[x1 y1 x2 y2]';
     yold=y0;
     % solve the systems of odes until next frame
     [t,y]=ode45('doublependulum2',[t0,t0+dt],y0,[],9.8,1,1,1,1);
     % keep the new angles and delete the old ones
     y0=y(size(y,1),:);
     clear y
     clear t
     % translate angles into [-pi,pi]
     y0(1)=mod(y0(1)+pi,2*pi)-pi;
     y0(3)=mod(y0(3)+pi,2*pi)-pi;
     t0=t0+dt;

     % delete the old pictures of the pendulum
     % calculate the new positions
```

```
        x1 =       L1*sin(y0(1));
        y1 =     - L1*cos(y0(1));
        x2 = x1 + L2*sin(y0(3));
        y2 = y1 - L2*cos(y0(3));
        % plot the path of the pendulumsin last time interval (overlay)
        plot([posold(1) x1],[posold(2) y1],'magenta');
        plot([posold(3) x2],[posold(4) y2],'black');
        % plot the images of the pendulums at new positions (overlay)
        set(b4,'xdata',[0,x1],'ydata',[0,y1]);
        set(b3,'xdata',[x1,x2],'ydata',[y1,y2]);
        set(b1,'xdata',x1+r1*bx,'ydata',y1+r1*by);
        set(b2,'xdata',x2+r2*bx,'ydata',y2+r2*by);
        drawnow
        pause(0.01)
    end

    figure(myfig);
    pause
    close(myfig)
```

System of equations

Doublependulum2.m

```
    %define a function that calculates the dynamics of the double pendulum
    %flag determines initial positions and velocities of the inner and outer
    %bob. Equations 14 to 17
    function xprime=doublependulum2(t,x,flag,g,l1,l2,m1,m2)
    xprime=zeros(4,1);
      xprime(1) = 6*(2*x(3)-3*cos(x(1)-x(2))*x(4))/(16-9*cos(x(1)-x(2))^2);
      xprime(2) = 6*(8*x(4)-3*cos(x(1)-x(2))*x(3))/(16-9*cos(x(1)-x(2))^2);
      xprime(3) = -(xprime(1)*xprime(2)*sin(x(1)-x(2))+3*g*sin(x(1)))/2;
      xprime(4) = -(-xprime(1)*xprime(2)*sin(x(1)-x(2))+g*sin(x(2)))/2;
    end
```

Calculating Energy of the function

Hamiltonian.m

```
    %function to calculate energy using equation 20
    function energy = hamiltonian(y0)
        t1 = y0(1);
        t2 = y0(2);
        v1 = y0(3);
        v2 = y0(4);
        g = 9.8;
        energy = abs(((3*(v2^2)-2*v1*v2*cos(t1-t2))/(2+2*(sin(t1-t2)^2)))-2*g*cos(t1)-
    g*cos(t2));
    end
```

Defining the constant surface for the Poincare section

Zerocross2.m

```
    function [v,ist,df] = zerocross2(t,x,varargin)% varargin absorbs unwanted
    parameters flag,g,l1, etc
    % event function for test_poincare.m

    v = sin(x(2)); % event is x(2)=0 (ie when v is zero)
    %v = x(2);
    ist = 1;  % if true, terminate evolution when this event occurs
    df = 1;   % increasing sense only
```

Producing the trajectory of the outer bob, Poincare section, angular velocity graph and energy of the system
PoincareTrajecAngleEnergy.m

```
clear
%Code mainly adapted from Alexander Barnett
%declare time step
T = 10;
%y0 = [0;0;-2;0];  % close to regular
%y0 = [pi;pi;0;0];
% y0 = [pi;pi;.5;0];  % chaotic
% y0=[0.5233;0;0.5233;0]%periodic
y0 = [0.2,0.2828,0,0]%perfect periodic with energy = 0.7809
%y0 = [0.2,-0.2828,0,0]%perfect quasiperiodic with energy = 0.7809
%y0 = [27.8;0;1.22;2.62]  %perfect KAM scenario
%y0 = [0,0,2,20]
%y0 = [0,0,1,20];
y0 =[pi;0;.5;.5];%high energy

s=ode45(@doublependulum2,[0,T],y0,[],0,1,1,1,1);
t = 0:0.01:T; x = deval(s,t)';
x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=x1+sin(x(:,2));
y2=y1-cos(x(:,2));

figure; plot(x2,y2); axis equal; xlabel('position');ylabel('velocity');title
('trajectory')
tmax = 1e2;       % max time to wait until next intersection
ns = 1000;         % how many intersection (iterations of P map)

tp = nan*(1:ns); yp = nan*zeros(numel(y0),ns); yi = y0; % init arrays
figure;
for n=1:ns
  s = ode45(@doublependulum2, [0, tmax], yi,
odeset('Events',@zerocross2,'abstol',1e-9),0,9.8,1,1,1,1);
  if isempty(s.xe), disp('no intersection found!');
  else tp(n) = s.xe(end); yi = s.ye(:,end); yp(:,n) = yi;
    %plot3(mod(yi(1),2*pi),yi(3),yi(4),'+'); hold on; if mod(n,10)==0, drawnow; end
% note it's in 3d now
    plot(yi(1),yi(4));hold on; if mod(n,10)==0, drawnow; end
  end
end
xlabel('x_1'); ylabel('x_3'); zlabel('x_4'); axis vis3d;
title('Poincare section (sin(x_2)=0, increasing)');

figure;
plot(t,x(:,3),'magenta',t,x(:,4),'black');title('Angular velocities');
energy = hamiltonian(y0)
```

Calculates the lyapunov exponent, the average lyapunov exponent and traces the separation of close trajectories
Crudelyap.m

```
clear
nruns =100;
%for calculating average later
h = zeros(1,nruns);
%indexing of h
k = 1;
for j=1:nruns
% y0 = [pi+0.1*rand(1);pi;.5;0];
```

```
% y0 = [0+0.1*rand(1);0;-2;0];

%y0 = [pi;pi;10.515;-2.17];%periodic with T = 0.5
%y0 = [];%quasiperiodic;
y0 = [pi+0.1*rand(1);pi;0.5;0];%chaotic
%y0 = [pi;pi;0;0];
%y0 = [0;0;-2;0];%close to regular
%y0 = [27.8;0;1.22;2.62];
%y0 = [pi;0;0.5;0.5];
%y0 = [0;0;2;20];

%y0 = [pi;0;5;5];%high energy
%y0 = [0.2;0.2828;0;0];%perfect periodic with energy = 0.7809
%y0 = [0.2+0.1*rand(1);-0.2828;0;0];
%y0 = [0.2;-0.2828;0;0];%perfect quasiperiodic with energy = 0.7809
T = 20;

o = odeset('abstol',1e-14);
s=ode45(@doublependulum2,[0,T],y0,o,0,1,1,1,1); % note changes
eps = 1e-9;
s2 = ode45(@doublependulum2,[0,T],y0+[eps;0;0;0],o,0,1,1,1,1); % note changes
t = 0:0.01:T; x = deval(s,t)';xe = deval(s2,t)';

x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=x1+sin(x(:,2));
y2=y1-cos(x(:,2));
xe1 = sin(xe(:,1));
ye1=-cos(xe(:,1));
xe2 = xe1+sin(xe(:,2));
ye2= ye1-cos(xe(:,2));
%store the size
N = numel(ye2);

%figure; plot(t, [y2 ye2], '-');title('Trajectories of two intial points with a
distance 10^-9')
logsep = zeros(1,N);
%loop fills matrix logsep with natural log of the changing separation
%between the initial trajectories
for i = 1:N
    logsep(i) = (log(abs(y2(i)-ye2(i)))-log(eps));
end
%figure; plot(t,logsep);title('lyapunov exponent');
t1=5;
t2=0.7;
%stores the exponent in a matric
h(k) = (logsep(t1/0.01)-logsep(t2/0.01))/(t1-t2);
k=k+1;
end
figure;
%average lyapunov exponent
plot(1:j,h)
title('average lyapunov exponent');
```

Graphs corresponding to the ‘other informative graphs’ sections of the input variables plotted against each other
Equilibria.m

```
clear
T = 10;
%y0 = [0;0;0;10]; %really good
%y0 = [pi;pi;0.5;0];
```

```
%y0 = [0;0;-2;0];
%y0=[0;0;2;20];
%y0 = [27.8;0;1.22;2.62]  %perfect KAM scenario
y0 = [0.2,-0.2828,0,0];
y0 = [pi;0;.5;.5]
%y0 = [pi;pi;0.5;0]
s=ode45(@doublependulum2,[0,T],y0,[],0,1,1,1,1);
t = 0:0.01:T; x = deval(s,t)';
%find energy for the inital conditions
energy = hamiltonian(y0)

%creates a plot with four figures of the variables plotted against each
%other
figure;
subplot(222)
plot(x(:,3),x(:,4));title('angular velocities')
subplot(221)
plot(x(:,1),x(:,2));title('angles')
subplot(223)
plot(x(:,1),x(:,3));title('positions and momentum of 1')
subplot(224)
plot(x(:,2),x(:,4));title('positions and momentum of 1')
```

Endnotes and References

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