Math 50: Midterm 2—practise questions

February 18, 2006

I cannot guarantee these questions cover all the material that we covered from 4.3 to 5.8. Details not covered include: $1/(n-1)$ estimate for $\hat{\sigma^2}$ of normal, ML estimate for uniform, failure of C-R bounds, Chebyshev and its failure. It is also about twice the actual length, but the questions are there to practise from. Use your HW7 first half, and old HW4-6 too!

1. Babe Ruth batted .356, *i.e.* 192 hits out of 540 bats. Based on that performance, construct a 95% confidence interval on probability of getting a future hit.

2. How large a sample n would you need to get a 95% confidence interval width of 3% in a yes/no survey sampled at random? Why doesn't your answer need to depend on the underlying p?

3. Samples drawn from $f_Y(y : \theta) = \theta y^{\theta-1}$ for $0 < y < 1$, with parameter $\theta > 0$.

   (a) Find ML estimator for θ.

   (b) Find (correctly-normalized) posterior pdf on θ if prior is uninformative. Harder one. [Hint: use $y^\theta = e^{\theta \ln y}$ to simplify]

   (c) find the Method of Moments estimator for θ.

4. In some countries (for cultural reasons) the probability of having a female child is 0.45. Estimate the probability of at least half of 1000 random children in this country being girls.

5. Demonstrate the ML estimator for p in a geometric pdf is $1/\bar{k}$ where $\bar{k}$ is the sample mean.

6. n samples are drawn from the uniform pdf on $0 < y < \theta$.

   (a) construct unbiased estimator for θ starting from Y_{min}.

   (b) Prove whether this estimator is consistent or not. (harder).

7. The mean μ of n data samples from a Normal pdf with known σ is to be estimated. Calculate the Cramer-Rao lower bound on the variance of any estimator. Compare this to the variance of the ML estimate which is the sample mean $\overline{Y}$.

8. (a) Show $\overline{Y}$ is an unbiased estimator of θ in the model pdf $f_Y(y; \theta) = (1/\theta)e^{-y/\theta}$.

   (b) Find the relative efficiency of this estimator to just using a single sample Y_1 as an estimator.

9. A coin of unknown bias is flipped twice and comes up with heads both times.

   (a) find the posterior on p using an uninformative prior.

   (b) repeat but using beta prior with $r = 2, s = 2$.

   (c) By finding area under posterior in the former case, find the Bayesian answer to the question, what is the chances that the coin has a bias less than 0.3, given the data?

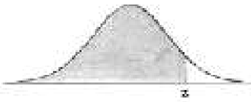
10. (a) You arrive at a bus stop. What is the pdf on the wait time to the arrival of the 2^{nd} bus, if on average they come every 10 minutes, at completely random times?

    (b) Find the chances of *exactly 2 buses* coming in the first 10 minutes.

    (c) Find the chances of the second bus coming in the first 10 minutes. [Hint: think carefully about the difference!].

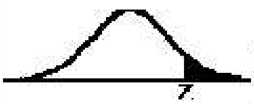
Useful formulae and pdfs:

$$
\begin{aligned}
f_{Y_i'}(y) &= \frac{n!}{(i-1)!(n-i)!} F_Y(y)^{i-1}[1-F_Y(y)]^{n-i} f_Y(y) \\
f_W(w) &= \int f_X(x) f_Y(w-x)dx \qquad \text{for} \quad W = X+Y \\
f_W(w) &= \int \frac{1}{|x|} f_X(w/x) f_Y(x)dx \qquad \text{for} \quad W = XY \\
f_W(w) &= \int |x| f_X(x) f_Y(wx)dx \qquad \text{for} \quad W = Y/X \\
\text{poisson } p_X(k;\lambda) &= e^{-\lambda}\frac{\lambda^k}{k!}, \text{ for } k = 0,1,2,\ldots, \quad \lambda \geq 0, \quad E(X) = \mathrm{Var}(X) = \lambda \\
\text{gamma } f_Y(y;r,\lambda) &= \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y}, \text{ for } y \geq 0, \qquad E(Y) = \frac{r}{\lambda}, \quad \mathrm{Var}(Y) = \frac{r}{\lambda^2} \\
\text{beta } f_Y(y;r,s) &= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} y^{r-1}(1-y)^{s-1} \text{ for } 0 \leq y \leq 1, \quad E(Y) = \frac{r}{r+s} \\
\text{normal } f_Y(y;\mu,\sigma) &= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad E(Y) = \mu, \quad \mathrm{Var}(Y) = \sigma^2 \\
\text{negative binomial } p_X(k;r,p) &= \binom{k-1}{r-1} p^k (1-p)^{k-r}, \text{ for } k = r, r+1, \ldots, \quad E(X) = \frac{r}{p}, \quad \mathrm{Var}(X) = \frac{r(1-p)}{p^2}
\end{aligned}
$$



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990



Far Right Tail Probabilities

Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}
2.0	0.02275	3.0	0.001350	4.0	0.00003167	5.0	2.867 E-7
2.1	0.01786	3.1	0.0009676	4.1	0.00002066	5.5	1.899 E-8
2.2	0.01390	3.2	0.0006871	4.2	0.00001335	6.0	9.866 E-10
2.3	0.01072	3.3	0.0004834	4.3	0.00000854	6.5	4.016 E-11
2.4	0.00820	3.4	0.0003369	4.4	0.000005413	7.0	1.280 E-12
2.5	0.00621	3.5	0.0002326	4.5	0.000003398	7.5	3.191 E-14
2.6	0.004661	3.6	0.0001591	4.6	0.000002112	8.0	6.221 E-16
2.7	0.003467	3.7	0.0001078	4.7	0.000001300	8.5	9.480 E-18
2.8	0.002555	3.8	0.00007235	4.8	7.933 E-7	9.0	1.129 E-19
2.9	0.001866	3.9	0.00004810	4.9	4.792 E-7	9.5	1.049 E-21

Answers

1. $(0.316, 0.396)$
2. $n \geq 4269$. indep of p since used worst case $p = 1/2$.
3. a) $\hat{\theta} = -n/\sum_i \ln y_i$, b) $g(\theta|y) = \frac{(\sum_i \ln y_i)^n}{\Gamma(n+1)} \cdot \theta^n e^{\theta \sum_i \ln y_i}$, just a gamma, c) $\bar{y}/(1-\bar{y})$.
4. $F_Z(-3.16) = 0.0008$ roughly, from table.
5. book p.348–9.
6. a) $\hat{\theta} = (n+1)Y_{min}$, b) $f_{\hat{\theta}}(u) \to (1/\theta)e^{-u/\theta}$ in the limit, which is a const pdf which doesn't converge on θ, so, no.
7. CR bound is σ^2/n, same as for sample mean estimator.
8. a) book 5.5.2 (see HW6), b) variance ratio is $1/n$.
9. a) $3\theta^2$, *i.e.* beta pdf with $r = 3, s = 1$, b) now has $r = 4, s = 2$, c) $(0.3)^3$
10. a) gamma with $r = 2$, $\lambda = 0.1$. b) Poisson with $\lambda T = 1$, $k = 2$, is $e^{-1}/2!$, c) definite integral gives $1 - 2e^{-1}$.