Math 50: Midterm 1

65 minutes, 70 points, no algebra-capable calculators. Show working/reaonsing, since only that way could you get partial credit. In the multiple-part questions, later parts are usually independent of earlier ones, so skip over one you can't do and come back later.

1. [16 points] An urn contains 4 red and 6 white chips. 5 chips are drawn at random, without replacement.
   (a) What is the probability that the first three chips follow exactly the sequence RWR?

   (b) What is the probability that the second chip is red?

   (c) What is the probability that the first chip was red given that the second chip is red?

   (d) What is probability of drawing a total of 2 red chips out of the 5?

   (e) Answer the previous question if they are drawn *with replacement*.

(f) What is the variance of the number of red chips drawn if they are drawn *with replacement*?

2. [8 points] Your burglar alarm is 99% reliable (if someone is breaking into your house, this is the chance of it going off). However there is a 1% chance of it going off on a given night when there's no break-in. Police estimate that break-ins occur at a given house about 1 in 1000 nights. If you hear the alarm, what's the chances there's a break-in?

3. [16 points] Random variables X and Y are sampled from the joint pdf $f_{X,Y}(x,y) = c(2x + y)$, for $0 \leq X \leq 1$ and $0 \leq Y \leq 1$, for some constant c.

(a) Find c.

(b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

(c) What is the probability that Y exceeds X?

(d) Are X and Y independent? (Explain)

(e) Find the expected value of Y given that X takes the value 1.

4. [15 points] Each day you go to the Novack Cafe and buy (and eat) a bag of chips. According to the manufacturer, the weight (in ounces) of chips X in any bag is a random variable with pdf $f_X(x) = e^{-x}$, $x > 0$. You may leave your answers as formulae involving e if you wish. [Hint: $\int_0^\infty x^n e^{-x} dx = n!$].

   (a) Find the pdf of the total weight of chips you ate in 2 days.

(b) What is the expected total weight of chips eaten in 1 week (7 days)?

(c) What is the standard deviation of the total weight eaten in 1 week?

(d) What are the chances that the smallest bag that week will *exceed* 1 ounce?

(e) What is the pdf of the weight of the largest bag that week?

5. [15 points] In a small village there is a 1% probability of a birth occurring each day (assume this is independent from day to day, and constant).

   (a) What is the mean number of births per year? (365 days)

   (b) Use a Poisson distribution to approximate the probability that 2 or more babies are born in a given half-year period.

(c) Let Y be the time (measured continuously in units of years) to the next birth. What is $f_Y(y)$?

(d) The year after a chemical factory moves to town, no births are reported in an entire year. How concerned are you? Explain your reasoning. [Hint: how likely is this to happen presuming no change in underlying birth rate?]

Useful formulae:

$$
\begin{aligned}
F_{Y_i'}(y) &= \frac{n!}{(i-1)!(n-i)!} F_Y(y)^{i-1}[1-F_Y(y)]^{n-i} f_Y(y) \\
f_W(w) &= \int f_X(x) f_Y(w-x) dx \qquad \text{for} \quad W = X+Y \\
f_W(w) &= \int \frac{1}{|x|} f_X(w/x) f_Y(x) dx \qquad \text{for} \quad W = XY \\
f_W(w) &= \int |x| f_X(x) f_Y(wx) dx \qquad \text{for} \quad W = Y/X
\end{aligned}
$$