MATH 50 WORKSHEET: pdf of a sum (convolution)

Barnett 4/18/06

Answers –

Imagine you and your friend are driving on a road trip.

Random vars. (independent)
$$\begin{cases} X = \text{length of time before you fall asleep at wheel} \\ Y = \text{'' '' your friend falls asleep at wheel} \end{cases}$$

When first one of you falls asleep, the other wakes up & takes over (this doesn't keep repeating; you each drive once). Pdfs are (in hours)

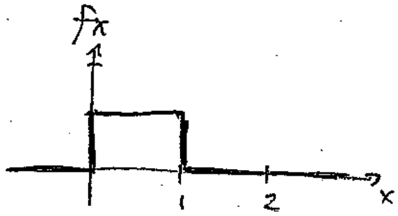
$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

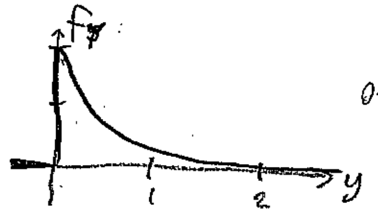
$$f_Y(y) = 2e^{-2y} \qquad y \geq 0$$

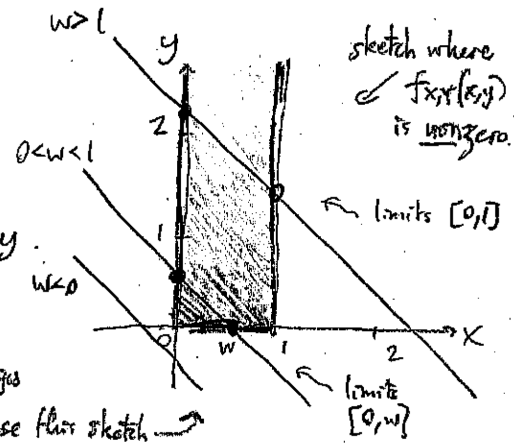
← your friend really is very tired!

What is pdf of W, total length of driving time?

First sketch







Careful: limits of integral may be different for w in different ranges

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

range $w<0$: diagonal line never hits nonzero region $\Rightarrow f_W(w) = 0$.

$0 \leq w < 1$: limits are $x \in [0,w]$ in order that integrand is nonzero.

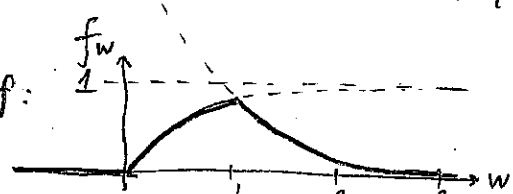
$$\int_0^w 1 \cdot 2e^{-2(w-x)}\,dx = 2e^{-2w}\int_0^w e^{2x}\,dx = 2e^{-2w}\cdot\tfrac{1}{2}e^{2x}\Big|_0^w$$

↑ note y substituted by $w-x$.

$$= e^{-2w}(e^{2w}-1) = 1 - e^{-2w}$$

$w>1$: $$\int_0^1 1\cdot 2e^{-2(w-x)}\,dx = e^{-2w}\underbrace{(e^2-1)}_{\approx 7}$$

Sketch the final pdf:



[Example 3.8.2 in book calls this 'cold redundancy'.]

keep convolving by f_X and f_Y.

Bonus: how would you calculate W if you kept alternately waking & sleeping? What would pdf look like after each done 5 sessions?