Math 42, Winter 2017

Homework set 1, due Wed Jan 11

This homework set is due on Wednesday January 11, at the start of class. Collaboration and discussion of the problems is permitted, and even recommended. But in the end you should write up your own solutions.

1. Suppose $f : (a, b) \to (c, d)$ is a smooth function that is surjective (onto), and $f'(x) \neq 0$ for all $x \in (a, b)$. Then either $f'(x) > 0$ for all $x \in (a, b)$, or $f'(x) < 0$ for all $x \in (a, b)$. This means that if $a < x_0 < x_1 < b$ then either always $f(x_0) < f(x_1)$ or always $f(x_0) > f(x_1)$. In both cases f is injective (one-to-one), and therefore f has an inverse function $g = f^{-1} : (c, d) \to (a, b)$.

   It is a basic fact from calculus that g has a derivative g' given by

   $$g'(y) = \frac{1}{f'(g(y))}$$

   A version of the *Inverse Function Theorem* that we used in class is as follows.

   Theorem 1 *If $f : (a, b) \to (c, d)$ is a smooth function that is surjective (onto), and $f'(x) \neq 0$ for all $x \in (a, b)$, then the inverse function $g = f^{-1} : (c, d) \to (a, b)$ is smooth.*

   A full proof of this theorem is discussed in a course on differential topology. In this problem you will prove that g has at least 3 derivatives.

   (a) Using the calculus formula for $g'(y)$ plus the chain rule, prove that $g''(y)$ exists.

   (b) Using the formula for $g''(y)$ that you found in item (a), prove that $g'''(y)$ exists.

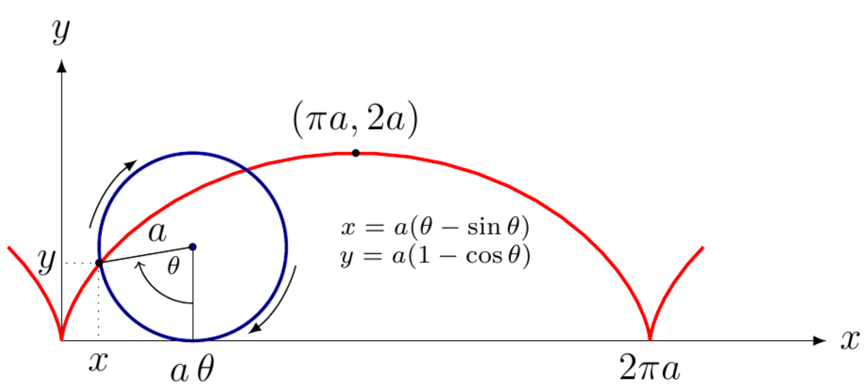
2. If you attach a dot to the circumference of a rolling wheel (circle), then the trajectory of this dot is a curve in $\mathbb{R}^2$ called a *cycloid*. If a is the radius of the circle, then a parametrization of the cycloid is

   $$\gamma(t) = \langle at - a \sin t,\ a - a \cos t\rangle$$

   (a) What are the regular points of γ?

   (b) Calculate the length of the arc of γ from $\gamma(0)$ to $\gamma(2\pi)$.

   (c) Does there exist a unit speed reparametrization $\tilde{\gamma}$ of γ? Explain your answer.



3. A straight line in $\mathbb{R}^n$ is a parametrized curve of the form $\gamma(t) = \mathbf{a} + t\,\mathbf{v}$, with $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$. Show that the length of the arc of a straight line between two points $\gamma(t_0)$ and $\gamma(t_1)$ is equal to the Euclidean distance between these two points in $\mathbb{R}^n$.

4. For each of the two curves γ below, find a unit speed reparametrization $\tilde{\gamma}(u)$ of $\gamma(t)$. Give explicit formulas for the reparametrization function $t = \phi(u)$ as well as the unit speed curve $\tilde{\gamma}(u)$ and the new parameter domain $(\tilde{\alpha}, \tilde{\beta})$ of $\tilde{\gamma}$.

   (a) The helix $\gamma(t) = \langle a\cos t, a\sin t, bt\rangle$ with $t \in (0, 2\pi)$.

   (b) The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(t) = \langle 2t, e^t + e^{-t}\rangle$.