23. Verify that the transformation rules

   (a) $\mathcal{F}(x \cdot T) = \left(\frac{-1}{2\pi i}\right)(\mathcal{F}T)'$; $\mathcal{F}(x^n \cdot T) = \left(\frac{-1}{2\pi i}\right)^n (\mathcal{F}T)^{(n)}$ and

   (b) $\mathcal{F}(e^{2\pi i a x} \cdot T) = \mathcal{F}T(s-a)$

   hold for any distribution T.

24. Find $\mathcal{F}(\cos 2\pi x)$ by expressing $\cos 2\pi x$ in terms of complex exponentials and using (b) from the previous problem. (The answer is on page A-5.)

25. Verify that

$$(\delta(7x))' = 7\delta'(7x)$$

   by

   (a) applying both sides to a test function, and

   (b) showing that both sides have the same Fourier transform. You may assume that all the transform rules on page A-13 hold for distributions, as well as for functions.

26. If α is a function and T is a distribution, use the definition

$$\langle \alpha * T, \varphi \rangle = \langle T, \widetilde{\alpha} * \varphi \rangle$$

   of their convolution to show that $\mathcal{F}(\alpha * T) = \mathcal{F}\alpha \cdot \mathcal{F}T$. *Hint:* Start with $\langle \mathcal{F}(\alpha * T), \varphi \rangle$, move all the operations to φ, then write $\widetilde{\alpha}$ as $\mathcal{F}\mathcal{F}^{-1}\widetilde{\alpha}$ and use the fact that $\mathcal{F}$ takes products to convolutions.

27. Express the *functions*

$$\delta(x-a) * \varphi$$

   and

$$(\delta(x-a)) + \delta) * \varphi,$$

   where φ is a test function and a is a constant, in as simple a form as possible. Your answers should not contain any distributions.

28. Find a solution to the DE

$$y'' + 8y' + 25y = f(t).$$

   Express your answer in terms of a convolution of functions and also as an integral.

29. Compute $Var(g) = \int_{\infty}^{\infty} |x|^2 |g(x)|^2 \, dx$ for the Gaussian $g(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2}$. Also compute $Var(g)Var(\mathcal{F}g)$. Could the constant $\frac{1}{16\pi^2}$ in Heisenberg's inequality be any larger?

30. The Gabor transform of f is defined by

$$\mathcal{G}f(s,m) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} e^{-(x-m)^2/2} \, dx.$$

   Show that

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{G}f(s,m) e^{2\pi i s x} e^{-(x-m)^2/2} \, ds \, dm \, .$$

Hint: The expression just above for $f(x)$ is a triple integral. Recognize the innermost integral with respect to u (assuming the name of the variable of integration in the definition of $\mathcal{G}f$ has been changed from x to u) as a Fourier transform, and the integral with respect to s as an inverse Fourier transform.

31.

(a) From class or page 420, the two distributions $\mathcal{F}(e^{i\pi x^2})$ and $e^{-i\pi s^2}$ satisfy

$$\mathcal{F}(e^{i\pi x^2}) = ce^{-i\pi s^2}$$

for some constant c. The value of c can be determined by applying these distributions to a test function–a convenient one is $\varphi(x) = e^{-\pi x^2}$. Find c by first expressing both sides of the equation

$$\langle \mathcal{F}(e^{i\pi x^2}), e^{-\pi s^2} \rangle = \langle e^{i\pi x^2}, \mathcal{F}(e^{-\pi s^2}) \rangle$$

as an integral. This gives the second line of Exercise 7.45 on p. 466. Then do the rest of Exercise 7.45.

(b) What are $\mathcal{F}^{-1}(e^{-i\pi s^2})$ and $\mathcal{F}(e^{-i\pi x^2})$?

(c) Use the dilation rule on p. A-13 to find $\mathcal{F}(e^{-i\pi a x^2})$ where a is any real constant. Consider the cases $a < 0$, $a = 0$ and $a > 0$ separately.

32. Find the solution $u(x,t)$ to the heat equation

$$\alpha^2 u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0$$

which satisfies the initial condition

$$u(x,0) = \delta(x-a), \quad -\infty < x < \infty$$

where a is a constant. Simplify your answer as much as possible.

33. Assume that f is a function satisfying $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ and $\int_{-\infty}^{\infty} |f(x)|^2\, dx = 1$. Show that if u is the solution to the free Schrödinger equation

$$u_t = \frac{i\lambda}{4\pi} u_{xx}$$

satisfying

$$u(x,0) = f(x),$$

then the integral

$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx = 1$$

for each value of $t > 0$. Hint: Do not work directly with u. Use the Parseval or Plancherel identity and work with $U = \mathcal{F}u$.

34. (Divergence theorem review problem) Evaluate the integral

$$\int_{\partial R} (x^2\mathbf{i} - 2xy\mathbf{j}) \cdot \mathbf{n}\, ds$$

where R is the region enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$ and $\mathbf{n}$ is the outward-pointing unit normal vector to the boundary ∂R of R.

35. Find the solution $u(x, y)$ to Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty, \quad y > 0$$

on the upper half-plane which satisfies the boundary condition

$$u(x, 0) = H(x) = \text{Heaviside function}, \quad -\infty < x < \infty.$$

Describe the level curves of u. What is the value of u along each of them? (Hint: The solution u can be expressed in a simple form in terms of the angle $\theta = \tan^{-1}\frac{y}{x}$ of polar coordinates.)

36. Find the solution to the non-homogeneous wave equation

$$c^2 u_{xx}(x, t) + f(x, t) = u_{tt}(x, t), \quad -\infty < x < \infty, \quad t > 0$$

satisfying the homogeneous initial conditions

$$u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty.$$

Write the solution in as simple a form as possible. (Hint: This is very similar in some ways to the problem

$$\alpha^2 u_{xx} + f(x, t) = u_t, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = 0, \quad -\infty < x < \infty$$

which we solved in class. Also, the forms of the answers to both problems are similar.)