Review for Finale

1. From the book Page 288 Ex. 2,3,4

2. Determine the values of a, b, c, d and e that minimize the integral:

$$\int_{-1}^{1}(x^5 - ax^4 - bx^3 - cx^2 - dx - e)^2 dx$$

Solution:
We have to approximate x^5 by a polynomial of degree 4 on the intervale $[-1, 1]$ with inner produce $< f, g >= \int_{-1}^{1} f(x)g(x)dx$ we know that the Legendre polynomial are orthonormal system in this intervale and

$$L_5 = \frac{1}{n^5 5!}\frac{d^5}{dx^5}(x^2 - 1)^5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

So:

$$x^5 = \frac{1}{63}(P_5(x) + 70x^3 - 15x)$$

and the best approximation of x^5 by a polynomial of degree less then 5 is $\frac{70}{63}x^3 - \frac{15}{63}$ therefore $a = c = e = 0$ and $b = \frac{70}{63}$ $d = -\frac{15}{63}$

3. Find the complex form of the Fourier series of the following functions:

(a)

$$f(x) = cosh(ax) \quad -\pi < x < \pi$$

Solution:

$$f(x) = \frac{sinh(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(1-)^n a}{a^2 + n^2}e^{inx}$$

(b)

$$f(x) = \cos(ax) \quad -\pi < x < \pi$$

Solution:

$$f(x) = \frac{\sin(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(1-)^n a}{a^2 - n^2}e^{inx}$$

(c)

$$f(x) = \cos(2x) + 3\cos(3x) \quad -\pi < x < \pi$$

Solution:

$$f(x) = e^{-3ix} + \frac{1}{2}e^{-2ix} + \frac{1}{2}e^{2ix} + e^{3ix}$$

4. Use D'Alembert's method to solve:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \leq x \leq 1 \qquad 0 < t$$

$$u(0,t) = 0 \qquad u(1,t) = 0$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial}(x,0) = g(x)$$

(a)

$$f(x) = \sin(\pi x) + 3\sin(2\pi x), \qquad g(x) = \sin(\pi x),$$

Solution:

$$u(x,t) = \frac{1}{2}[\sin(\pi(x-t)) + \sin(\pi(x+t)) + 3\sin(2\pi(x-t)) + 3\sin(\pi(x+t)) +$$

$$\frac{1}{2\pi}[\cos(\pi(x-t) - \cos(\pi(x+t))]]$$

(b)

$$f(x) = 0, \qquad g(x) = -10,$$

Solution:

$$u(x,t) = \frac{1}{2}[G(x+t) - G(x-t)]$$

Where $G(x)$ in 2-periodic function and:

$$G(x) = \begin{cases} -10x & 0 < x < 1 \\ 10x - 20 & 1 < x < 2 \end{cases}$$

5. solve:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t} \qquad -\pi \leq x \leq \pi \qquad 0 < t$$

$$u(-\pi, t) = u(\pi, t), \qquad u_x(-\pi, t) = u_x(\pi, t)$$
$$u(x, 0) = |x|$$

Solution:

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{(n odd)}}}^{\infty} \frac{1}{n^2} e^{-a^2 n^2 t} \cos(nx)$$

6. solve:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u \qquad 0 \leq x \leq \pi \qquad 0 < t$$

$$u_x(0, t) = 0, \qquad u_x(\pi, t) = 0$$
$$u(x, 0) = x^2$$

Solution:

$$u(x,t) = \frac{\pi^2}{3} e^{-t} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(1+3n^2)t} \cos(nx)$$

7. Find the Fourier transform of:

   (a)

$$f(x) = \frac{\sin(ax)}{x}$$

   Solution:

$$F(t) = \begin{cases} \sqrt{\frac{\pi}{2}} & |t| < a, \\ 0 & otherwise \end{cases}$$

(b)

$$f(x) = \frac{a - ix}{a^2 + x^2}$$

Solution:

$$F(t) = \begin{cases} \sqrt{2\pi} e^{-at} & t > 0, \\ 0 & otherwise \end{cases}$$

8. Solve:

$$t\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad -\infty < x < \infty \qquad 0 < t$$
$$u(x, 0) = f(x)$$

Solution:
Using Fourier transform method we get:

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-\frac{t^2 s^2}{2}} e^{isx} ds$$

9. Find the Laplace transform of:

(a)

$$\sqrt{t} + \frac{1}{\sqrt{t}}$$

Solution:

$$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} + \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

(b)

$$te^{-t} \sin(t)$$

Solution:

$$\frac{2(1+s)}{(1 + (1+s)^2)^2}$$

10. Find the inverse Laplace transform of:

$$\frac{2s-1}{s^2-s-2}$$

Solution:

$$e^{-t}+e^{2t}$$

11. Solve

$$\nabla u(r,\theta)=0 \qquad 0<r<\rho, \qquad -\pi<\theta<\pi$$
$$u(\rho,\theta)=\cos^2(\theta)$$

Solution:

$$u(r,\theta)=\frac{1}{2}[1+(\frac{r}{\rho})^2\cos(2\theta)]$$