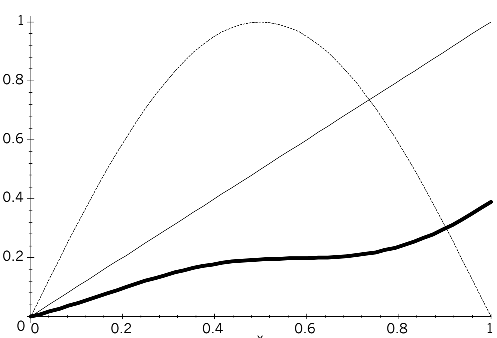
Mathematics 33

Homework #5

Due May 10

1. (p. 26, 2, 5 points). What is your interpretation of the initial-boundary-value problem? PDE: $u_t = \alpha^2 u_{xx}$, $0 < x < 1, t > 0$; BCs: $u(0,t) = 0, u_x(1,t) = 1, t > 0$; IC: $u(x,0) = \sin(\pi x)$, $0 \le x \le 1$. Can you draw rough sketches of the solution for different values of time? Will the solution come to a steady state; is this obvious?

Solution. The right boundary condition $u_x(1,t) = 1, t > 0$ means that the heat flows **in** with unit speed. Since the steady state solution $\overline{u} = \overline{u}(x)$ for PDE $u_t = \alpha^2 u_{xx}$ is a linear function of x it implies that $\overline{u}(x) = x$. Indeed, $\overline{u}(0) = 0$ and $\overline{u}_x(1) = 1$. The sketch of the temperature distribution at different time is shown below. Notice, the bold line has slope 1 at $x = 1$.



Dashed line – initial condition temperature ($t = 0$), solid – steady state solution ($t = \infty$), bold line – temperature for intermidiate time.

2. (p. 26, 4, 5 points). Suppose a metal rod laterally insulated has an initial temperature of 20^oC but immediately thereafter has one end fixed at 50^oC. The rest of the rod is immersed in a liquid solution temperature 30^oC. What would be the IBVP that describes this problem.

Solution. If L is the length of the rod than IC is $u(x,0) = 20, 0 \le x \le L$. Let the temperature of the right end is kept at 50^oC that implies BC $u(L,t) = 50$ and for the left end we have $u(0,t) = 30$

for all time $t > 0$.

3. (p. 31, 1, 5 points). Substitute the units of each quantity $u, u_t, ...$ into the equation $u_t = \alpha^2 u_{xx} - \beta u$ to see that every term has the same units of $^oC/\sec$.

Solution. u_t measures the speed at which temperature changes with time, i.e. the units are $^oC/\sec$. Let us assume the length of the rod is measured in cm. Then u_{xx} is measured in $^oC/cm^2$ and α^2 has units $\text{cm}^2/\sec$ so that $\alpha^2 u_{xx}$ is measured in $\text{cm}^2/\sec \times ^oC/cm^2 =^o C/\sec$. Parameter β must be measured in 1/sec in order to have $^oC/\sec$ for the last term, βu.

4. (p. 42, 5, 5 points). What is the solution to problem 4 if the IC is changed to

$$u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x).$$

Solution. The key point to the solution is the Fourier sine expansion of $\phi(x) = u(x,0)$. But since $u(x,0)$ is a linear combination of sine functions we can determine the Fourier coefficients without evaluating integrals, or more precisely, $n = 1, A_1 = 0$, $n = 2, A_2 = 1$, $n = 3, A_3 = 0$, $n = 4, A_4 = 1/3$, $n = 5, A_5 = 0$, $n = 6, A_6 = 1/5$, and $A_n = 0$ for $n > 6$. Therefore, the solution is $(\alpha = 1)$

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-(\alpha\pi n)^2 t}\sin(\pi n x) \\ &= e^{-4\pi^2 t}\sin(2\pi x) + \frac{1}{3}e^{-16\pi^2 t}\sin(4\pi x) + \frac{1}{5}e^{-36\pi^2 t}\sin(6\pi x). \end{aligned}$$