Mathematics 33

Homework Assignment #3

Due Wednesday, April 19

1. (4 points). Let $C[0,1]$ denote the set of all continuous functions $f(x), x \in [0,1]$ with the scalar product defined as $(f,g) = \int_0^1 f(x)g(x)dx$. Let $p_0(x) = 1$ be the null-degree polynomial on $x \in [0,1]$. Find the first-degree polynomial $p_1(x) = a + bx$ such that $p_0 \perp p_1$ and $\|p_1\| = 1$.

Solution. We find a and b from the conditions

$$(p_0, p_1) = \int_0^1 (a+bx)dx = 0, \ \|p_1\|^2 = \int_0^1 (a+bx)^2 dx = 1.$$

We have

$$\begin{aligned} \int_0^1 (a+bx)dx &= a + b/2 = 0, \\ \int_0^1 (a+bx)^2 dx &= \int_0^1 (a^2 + 2abx + b^2x^2)dx = a^2 + ab + b^2/3 = 1. \end{aligned}$$

From the first equation we have $b = -2a$ and substituting it into the second equation we obtain $a^2 - 2a^2 + 4a^2/3 = 1$ which yields $a = \pm\sqrt{3}$ and $b = -2\sqrt{3}$ if $a = \sqrt{3}$ and $b = 2\sqrt{3}$ if $a = -\sqrt{3}$ (two solutions). Finally, we have $p_1(x) = \pm\sqrt{3}(1-2x)$.

2. (5 points). Check that functions $p_0(x) = 1$ and $p_1(x) = x - \pi/4$ are orthogonal on $C[0, \pi/2]$. Find the best linear approximation of $f(x) = \sin x$ by p_0 and p_1 on $[0, \pi/2]$. Compute the squared norm of approximation.

Solution. To prove the orthogonality we need to show that $\int_0^{\pi/2}(x - \pi/4)dx = 0$. We have

$$\int_0^{\pi/2} (x - \pi/4)dx = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{\pi}{2}\frac{\pi}{4} = 0.$$

The best linear approximation for $f(x)$ is found in the form $\lambda_0 p_0(x) + \lambda_1 p_1(x)$ where

$$\lambda_0 = \frac{(f, p_0)}{\|p_0\|^2}, \ \lambda_1 = \frac{(f, p_1)}{\|p_1\|^2}.$$

We have

$$\begin{aligned} (f, p_0) &= \int_0^{\pi/2} \sin x dx = -\cos x|_0^{\pi/2} = 1, \quad \|p_0\|^2 = \int_0^{\pi/2} dx = \pi/2, \\ (f, p_1) &= \int_0^{\pi/2} (x - \pi/4)\sin x dx = \int_0^{\pi/2} x \sin x dx - \pi/4 \int_0^{\pi/2} \sin x dx = 1 - \pi/4, \\ \|p_1\|^2 &= \int_0^{\pi/2} (x - \pi/4)^2 dx = \pi^3/96. \end{aligned}$$

Thus,

$$\lambda_0 = \frac{2}{\pi}, \lambda_1 = \frac{1-\pi/4}{\pi^3/96},$$

and the best linear approximation for $\sin x$ on $[0, \pi/2]$ is

$$\widehat{f} = \lambda_0 p_0(x) + \lambda_1 p_1(x) = \frac{2}{\pi} + 96\frac{1-\pi/4}{\pi^3}(x - \frac{\pi}{4}) = 0.1477 + 0.66444x,$$

see the graph below. The squared norm of approximation is computed as

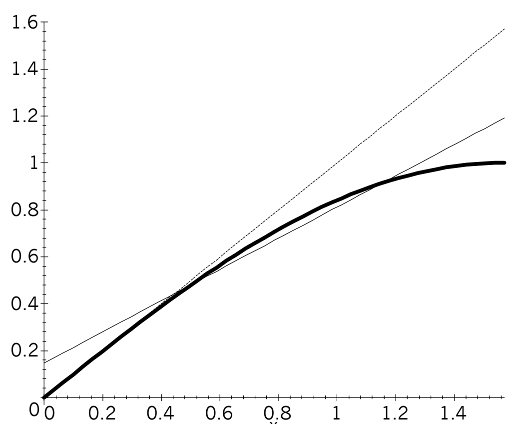
$$\begin{aligned} \int_0^{\pi/2} (\sin x - 0.1477 - 0.66444x)^2 dx &= \int_0^{\pi/2} \sin^2 x dx - \int_0^{\pi/2} (0.1477 + 0.66444x)^2 \, dx \\ &= 7.8919 \times 10^{-3}. \end{aligned}$$

3. (5 points). Find the first-order approximation of $\sin x$ at $x_0 = 0$ using Taylor series expansion and compute the squared norm of approximation using the scalar product $(f, g) = \int_0^{\pi/2} f(x)g(x)dx$.

Solution. Taylor series expansion of the first-order of $\sin x$ at $x_0 = 0$ gives $\sin x \simeq x$. The squared norm of approximation is

$$\int_0^{\pi/2} (x - \sin x)^2 dx = \int_0^{\pi/2} \sin^2 x dx - 2\int_0^{\pi/2} x \sin x dx + \int_0^{\pi/2} x^2 dx = \frac{1}{4}\pi - 2 + \frac{1}{24}\pi^3 = 7.7326 \times 10^{-2}$$

We notice that the error is larger than in the previous problem. It must be larger because the approximation in the previous problem provides the minimum of the error.



Approximation of $\sin x$ on $[0, \pi/2]$ by two linear functions. The first is the best linear approximation (solid) and the second is the first-order Taylor series expansion (dashed). The latter has a larger approximation error.

4. (7 points). Determine the Fourier series expansion of the function $f(x) = \pi^2 - x^2$ for $-\pi \le x \le \pi$. Compute the squared norm of approximation based on the first two terms of the Fourier series.

Solution. Since $f(x)$ is an even function $b_n = 0$. The constant term is

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)dx = \frac{4}{3}\pi^2.$$

The a_n term is

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\cos nx dx = \pi\int_{-\pi}^{\pi}\cos nx dx - \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx dx \\ &= \pi\left(\frac{1}{n}\sin nx\right)\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\left(\frac{2x\cos nx}{n^2} + \frac{(n^2x^2-2)\sin nx}{n^3}\right)\Big|_{-\pi}^{\pi} \\ &= (-1)^{n+1}\frac{4}{n^2}. \end{aligned}$$

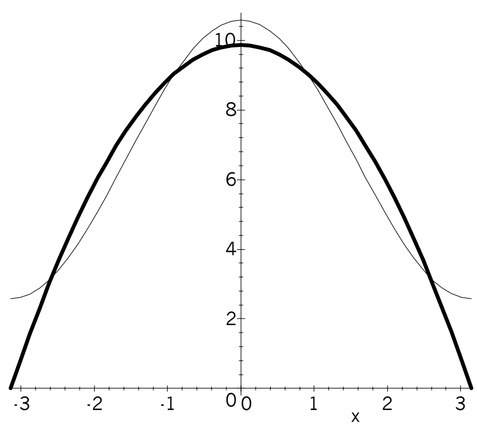
Finally, the Fourier series is

$$\pi^2 - x^2 = \frac{2}{3}\pi^2 + 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + ...\right)$$

The first two terms give the approximation

$$\pi^2 - x^2 \simeq \frac{2}{3}\pi^2 + 4\cos x,$$

plotted below.



Bold – function $\pi^2 - x^2$ on $[-\pi, \pi]$, solid – approximation based on the first two terms of Fourier series.

The squared norm of approximation is computed by the formula

$$\begin{aligned} &\int_{-\pi}^{\pi}(\pi^2 - x^2 - \frac{2}{3}\pi^2 - 4\cos x)^2 dx \\ = &\int_{-\pi}^{\pi}(\pi^2 - x^2)^2 dx - \pi\left(\frac{1}{2}a_0^2 + a_1^2\right) = \frac{16}{15}\pi^5 - \pi\left(\frac{16}{18}\pi^4 + 4^2\right) = 4.138. \end{aligned}$$