Chapter 4

Algebraic Counting Techniques

4.1 The Principle of Inclusion and Exclusion

4.1.1 The size of a union of sets

One of our very first counting principles was the sum principle which says that the size of a union of disjoint sets is the sum of their sizes. Computing the size of overlapping sets requires, quite naturally, information about how they overlap. Taking such information into account will allow us to develop a powerful extension of the sum principle known as the "principle of inclusion and exclusion."

141. In a biology lab study of the effects of basic fertilizer ingredients on plants, 16 plants are treated with potash, 16 plants are treated with phosphate, and among these plants, eight are treated with both phosphate and potash. No other treatments are used. How many plants receive at least one treatment? If 32 plants are studied, how many receive no treatment?

142. Give a formula for the size of the union $A \cup B$ of two sets A in terms of the sizes $|A|$ of A, $|B|$ of B, and $|A \cap B|$ of $A \cap B$. If A and B are subsets of some "universal" set U, express the size of the complement $U - (A \cup B)$ in terms of the sizes $|U|$ of U, $|A|$ of A, $|B|$ of B, and $|A \cap B|$ of $A \cap B$.

143. In Problem 141, there were just two fertilizers used to treat the sample plants. Now suppose there are three fertilizer treatments, and 15 plants

are treated with nitrates, 16 with potash, 16 with phosphate, 7 with nitrate and potash, 9 with nitrate and phosphate, 8 with potash and phosphate and 4 with all three. Now how many plants have been treated? If 32 plants were studied, how many received no treatment at all?

144. Give a formula for the size of $A_1 \cup A_2 \cup A_3$ in terms of the sizes of A_1, A_2, A_3 and the intersections of these sets.

145. Conjecture a formula for the size of a union of sets

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

in terms of the sizes of the sets A_i and their intersections.

The difficulty of generalizing Problem 144 to Problem 145 is not likely to be one of being able to see what the right conjecture is, but of finding a good notation to express your conjecture. In fact, it would be easier for some people to express the conjecture in words than to express it in a notation. Here is some notation that will make your task easier. Let us define

$$\bigcap_{i: i \in I} A_i$$

to mean the intersection over all elements i in the set I of A_i. Thus

$$\bigcap_{i: i \in \{1,3,4,6\}} = A_1 \cap A_3 \cap A_4 \cap A_6. \tag{4.1}$$

This kind of notation, consisting of an operator with a description underneath of the values of a dummy variable of interest to us, can be extended in many ways. For example

$$\begin{aligned}\sum_{I: I \subseteq \{1,2,3,4\},\ |I|=2} |\cap_{i \in I} A_i| &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\ &+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|. \end{aligned} \tag{4.2}$$

146. Use notation something like that of Equation 4.1 and Equation 4.2 to express the answer to Problem 145. Note there are many different correct ways to do this problem. Try to write down more than one and

choose the nicest one you can. Say why you chose it (because your view of what makes a formula nice may be different from somebody else's). The nicest formula won't necessarily involve all the elements of Equations 4.1 and 4.2.

147. A group of n students goes to a restaurant carrying backpacks. The manager invites everyone to check their backpack at the check desk and everyone does. While they are eating, a child playing in the check room randomly moves around the claim check stubs on the backpacks. What is the probability that, at the end of the meal, at least one student receives his or her own backpack? In other words, in what fraction of the total number of ways to pass the backpacks back does at least one student get his or her own backpack back? (It might be a good idea to first consider cases with $n = 3$, 4, and 5. Hint: For each student, how big is the set of backpack distributions in which that student gets the correct backpack?) What is the probability that no student gets his or her own backpack?

148. As the number of students becomes large, what does the probability that no student gets the correct backpack approach?

The formula you have given in Problem 146 is often called **the principle of inclusion and exclusion** for unions of sets. The reason is the pattern in which the formula first adds (includes) all the sizes of the sets, then subtracts (excludes) all the sizes of the intersections of two sets, then adds (includes) all the sizes of the intersections of three sets, and so on. Notice that we haven't yet proved the principle. We will first describe the principle in an apparently more general situation that doesn't require us to translate each application into the language of sets. While this new version of the principle might seem more general than the principle for unions of sets; it is equivalent. However once one understands the notation used to express it, it is more convenient to apply.

Problem 147 is sometimes called the *hatcheck problem*; the name comes from substituting hats for backpacks. If is also sometimes called the *derangement problem*. A *derangement* of an n-element set is a permutation of that set (thought of as a bijection) that maps no element of the set to itself. One can think of a way of handing back the backpacks as a permutation f of the students: $f(i)$ is the owner of the backpack that student i receives. Then a derangement is a way to pass back the backpacks so that no student gets his or her own.

4.1.2 The hatcheck problem restated

The last question in Problem 147 requires that we compute the number of ways to hand back the backpacks so that nobody gets his or her own backpack. We can think of the set of ways to hand back the backpacks so that student i gets the correct one as the set of permutations of the backpacks with the property that student i gets his or her own backpack. Since there are $n-1$ other students and they can receive any of the remaining $n-1$ backpacks in $(n-1)$ ways, the number of permutations with the property that student i gets the correct backpack is $(n-1)!$. How many permutations are there with the properties that student i gets the correct backpack *and* student j gets the correct backpack? (Let's call these properties i and j for short.) Since there are $n-2$ remaining students and $n-2$ remaining backpacks, the number of permutations with properties i and j is $(n-2)!$. Similarly, the number of permutations with properties $i_1, i_2, \ldots, i_k$ is $(n-k)!$. Thus when we compute the size of the union of the sets

$$S_i = \{f : f \text{ is a permutation with property } i\},$$

we are computing the number of ways to pass back the backpacks so that at least one student gets the correct backpack. This answers the first question in Problem 147. The last question in Problem 147 is asking us for the number of ways to pass back the backpacks that have *none* of the properties. To say this in a different way, the question is asking us to compute the number of ways of passing back the backpacks that have exactly the *empty set*, $\emptyset$, of properties.

4.1.3 Basic counting functions: $N_{\textbf{at least}}$ and $N_{\textbf{exactly}}$

Notice that the quantities that we were able to count easily were the number of ways to pass back the backpacks so that we satisfy a certain subset $K = \{i_1, i_2, \ldots, i_k\}$ of our properties. In fact, among the $(n-k)$ ways to pass back the backpacks with this particular set K of properties is the permutation that gives each student the correct backpack, and has not just the properties in K, but the whole set of properties. Similarly, for any set M of properties with $K \subseteq M$, the permutations that have all the properties in M are among the $(n-k)!$ permutations that have the properties in the set K. Thus we can think of $(n-k)!$ as counting the number of permutations that have *at least* the properties in K. In particular, $n!$ is the number of ways to pass back the backpacks that have at least the empty set of properties. We thus write $N_{\text{at least}}(\emptyset) = n!$, or $N_{\text{a}}(\emptyset) = n!$ for short. For a k-element subset K of the properties, we write $N_{\text{at least}}(K) = (n-k)!$ or $N_{\text{a}}(K) = (n-k)!$ for short.

The question we are trying to answer is "How many of the distributions of backpacks have exactly the empty set of properties?" For this purpose we introduce one more piece of notation. We use $N_{\text{exactly}}(\emptyset)$ or $N_{\text{e}}(\emptyset)$ to stand for the number of backpack distributions with exactly the empty set of properties, and for any set K of properties we use $N_{\text{exactly}}(K)$ or $N_{\text{e}}(K)$ to stand for the number of backpack distributions with exactly the set K of properties. Thus $N_{\text{e}}(K)$ is the number of distribution in which the students represented by the set K of properties get the correct backpacks back and no other students do.

4.1.4 The principle of inclusion and exclusion for properties

For the principle of inclusion and exclusion for properties, suppose we have a set of arrangements (like backpack distributions) and a set P of properties (like student i gets the correct backpack) that the arrangements might or might not have. We suppose that we know (or can easily compute) the numbers $N_{\text{a}}(K)$ for every subset K of P. We are most interested in computing $N_{\text{e}}(\emptyset)$, the number of arrangements with none of the properties, but it will turn out that with no more work we can compute $N_{\text{e}}(K)$ for every subset K of P. Based on our answer to Problem 147 we expect that

$$N_{\text{e}}(\emptyset) = \sum_{S: S \subseteq P} (-1)^{|S|} N_{\text{a}}(S) \tag{4.3}$$

and it is a natural guess that

$$N_{\text{e}}(K) = \sum_{S: K \subseteq S \subseteq P} (-1)^{|S|} N_{\text{a}}(S). \tag{4.4}$$

Equations 4.3 and 4.4 are called **the principle of inclusion and exclusion for properties**.

149. Verify that the formula for the number of ways to pass back the backpacks in Problem 147 so that nobody gets the correct backpack has the form of Equation 4.3.

150. Find a way to express $N_{\mathrm{a}}(K)$ in terms of $N_{\mathrm{e}}(S)$ for subsets of P containing K. In particular, what is the equation that expresses $N_{\mathrm{a}}(\emptyset)$ in terms of $N_{\mathrm{e}}(S)$ for subsets S of P?

151. Substitute the formula for N_{a} from Problem 150 into the right hand sides of the formulas of Equations 4.3 and 4.4 and simplify what you get to show that Equations 4.3 and 4.4 are in fact true. (Hint: the binomial theorem may help you simplify the formulas you get.)

152. In how many ways may we distribute k identical apples to n children so that no child gets more than three apples?

4.1.5 Counting onto functions

153. Given a function f from the k-element set K to the n-element set $[n]$, we say f has property i if $f(x) \neq i$ for every x in K. How many of these properties does an onto function have? What is the number of functions from a k-element set onto an n-element set?

154. Find a formula for the Stirling number (of the second kind) $S(k, n)$.

4.1.6 The chromatic polynomial of a graph

We defined a graph to consist of set V of elements called vertices and a set E of elements called edges such that each edge joins two vertices. A *coloring* of a graph by the elements of a set C (of colors) is an assignment of an element of C to each vertex of the graph; that is, a function from the vertex set V of the graph to C. A coloring is called *proper* if for each edge joining two distinct vertices[1], the two vertices it joins have different colors. You may have heard of the famous four color theorem of graph theory that says if a graph may be drawn in the plane so that no two edges cross (though they may touch at a vertex), then the graph has a proper coloring with four colors. Here we are

[1]If a graph had a loop connecting a vertex to itself, that loop would connect a vertex to a vertex of the same color. it is because of this that we only consider edges with two distinct vertices in our definition

interested in a different, though related, problem: namely, in how many ways may we properly color a graph (regardless of whether it can be drawn in the plane or not)h using k or fewer colors? When we studied trees, we restricted ourselves to connected graphs. (Recall that a graph is connected if, for each pair of vertices, there is a walk between them.) Here, disconnected graphs will also be important to us. Given a graph which might or might not be connected, we partition its vertices into blocks called *connected components* as follows. For each vertex v we put all vertices connected to it by a walk into a block together. Clearly each vertex is in at least one block, because vertex v is connected to vertex v by the trivial walk consisting of the single vertex v and no edges. To have a partition, each vertex must be in one and only one block. To prove that we have defined a partition, suppose that vertex v is in the blocks B_1 and B_2. Then B_1 is the set of all vertices connected by walks to some vertex v_1 and B_2 is the set of all vertices connected by walks to some vertex v_2.

155. Show that $B_1 = B_2$.

Since $B_1 = B_2$, these two sets are the same block, and thus all blocks containing v are identical, so v is in only one block. Thus we have a partition of the vertex set, and the blocks of the partition are the connected components of the graph. Notice that the connected components depend on the edge set of the graph. If we have a graph on the vertex set V with edge set E and another graph on the vertex set V with edge set E', then these two graphs could have different connected components. It is traditional to use the Greek letter γ (gamma)[2] to stand for the number of connected components of a graph; in particular, $\gamma(V, E)$ stands for the number of connected components of the graph with vertex set V and edge set E. We are going to show how the principle of inclusion and exclusion may be used to compute the number of ways to properly color a graph using colors from a set C of c colors.

156. Suppose we have a graph G with vertex set V and edge set E. Suppose F is a subset of E. Suppose we have a set C of c colors with which to color the vertices.

    (a) In terms of $\gamma(V, F)$, in how many ways may we color the vertices of g so that each edge in F connects two vertices of the same color?

[2]The greek letter gamma is pronounced gam-uh, where gam rhymes with ham.

(b) Given a coloring of G, for each edge e in E, let us consider the property that the endpoints of e are colored the same color. Let us call this property "property e." In this way each set of properties can be thought of as a subset of E. What set of properties does a proper coloring have?

(c) Find a formula (which may involve summing over all subsets F of the edge set of the graph and using the number $\gamma(V, F)$ of connected components of the graph with vertex set V and edge set F) for the number of proper colorings of G using colors in the set C.

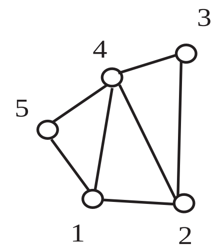
The formula you found in Problem 156c is a formula that involves powers of c, and so it is a polynomial function of c. Thus it is called the *chromatic polynomial* of the graph G. Since we like to think about polynomials as having a variable x and we like to think of c as standing for some constant, people often use x as the notation for the number of colors we are using to color G. Frequently people will use $\chi_G(x)$ to stand for the number of way to color G with x colors, and call $\chi_G(x)$ the chromatic polynomial of G.

157. In Chapter 2 we introduced the deletion-contraction recurrence for counting spanning trees of a graph. Figure out how the chromatic polynomial of a graph is related to those resulting from deletion of an edge e and from contraction of that same edge e. Try to find a recurrence like the one for counting spanning trees that expresses the chromatic polynomial of a graph in terms of the chromatic polynomials of $G - e$ and G/e for an arbitrary edge e. Use this recurrence to give another proof that the number of ways to color a graph with x colors is a polynomial function of x.

158. Use the deletion-contraction recurrence to compute the chromatic polynomials of the graph in Figure 4.1. (You can simplify your computations by thinking about the effect on the chromatic polynomial of deleting an edge that is a loop, or deleting one of several edges between the same two vertices.)

159. In how many ways may you properly color the vertices of a path on n vertices with x colors? Describe any dependence of the chromatic polynomial of a path on the number of vertices. In how many ways may you properly color the vertices of a cycle on n vertices with x

Figure 4.1: A graph.



colors? Describe any dependence of the chromatic polynomial of a cycle on the number of vertices.

160. In how many ways may you properly color the vertices of a tree on n vertices with x colors?

161. What do you observe about the signs of the coefficients of the chromatic polynomial of the graph in Figure 4.1? What about the signs of the coefficients of the chromatic polynomial of a path? Of a cycle? Of a tree? Make a conjecture about the signs of the coefficients of a chromatic polynomial and prove it.

4.2 The Idea of Generating Functions

Suppose you are going to choose three pieces of fruit from among apples, pears and bananas for a snack. We can symbolically represent all your choices as

🍎🍎🍎 + 🍐🍐🍐 + 🍌🍌🍌 + 🍎🍎🍐 + 🍎🍎🍌 + 🍎🍐🍐 + 🍐🍐🍌 + 🍎🍌🍌 + 🍐🍌🍌 + 🍎🍐🍌.

Here we are using a picture of a piece of fruit to stand for taking a piece of that fruit. Thus 🍎 stands for taking an apple, 🍎🍐 for taking an apple and a pear, and 🍎🍎 for taking two apples. You can think of the plus sign as standing for the "exclusive or," that is, 🍎 + 🍌 would stand for "I take an apple or a banana but not both." To say "I take both an apple and a banana," we would write 🍎🍌. We can extend the analogy to mathematical notation by condensing our statement that we take three pieces of fruit to

🍎^{3} + 🍐^{3} + 🍌^{3} + 🍎^{2}🍐 + 🍎^{2}🍌 + 🍎🍐^{2} + 🍐^{2}🍌 + 🍎🍌^{2} + 🍐🍌^{2} + 🍎🍐🍌.

In this notation 🍎^{3} stands for taking a multiset of three apples, while 🍎^{2}🍌 stands for taking a multiset of two apples and a banana, and so on. What our notation is really doing is giving us a convenient way to list all three element multisets chosen from the set {🍎, 🍐, 🍌}.

Suppose now that we plan to choose between one and three apples, between one and two pears, and between one and two bananas. In a somewhat clumsy way we could describe our fruit selections as

🍎🍐🍌 + 🍎^{2}🍐🍌 + $\cdots$ + 🍎^{2}🍐^{2}🍌 + $\cdots$ + 🍎^{2}🍐^{2}🍌^{2} + 🍎^{3}🍐🍌 + $\cdots$ + 🍎^{3}🍐^{2}🍌 + $\cdots$ + 🍎^{3}🍐^{2}🍌^{2}. (4.5)

162. Using an A in place of the picture of an apple, a P in place of the picture of a pear, and a B in place of the picture of a banana, write out the formula similar to Formula 4.5 without any dots for left out terms. (You may use pictures instead of letters if you prefer, but it gets tedious quite quickly!) Now expand the product $(A+A^2+A^3)(P+P^2)(B+B^2)$ and compare the result with your formula.

163. Substitute x for all of A, P and B (or for the corresponding pictures) in the formula you got in Problem 162 and expand the result in powers of x. Give an interpretation of the coefficient of x^n.

If we were to expand the formula

(🍎 + 🍎^{2} + 🍎^{3})(🍐 + 🍐^{2})(🍌 + 🍌^{2}). (4.6)

we would get Formula 4.5. Thus formula 4.5 and formula 4.6 each describe the number of multisets we can choose from the set 🍎, 🍐, 🍌 in which 🍎 appears between 1 and three times and 🍐, and 🍌 each appear once or twice. We interpret Formula 4.5 as describing each individual multiset we can choose, and we interpret Formula 4.6 as saying that we first decide how many apples to take, and then decide how many pears to take, and then decide how many bananas to take. At this stage it might seem a bit magical that doing ordinary algebra with the second formula yields the first, but in fact we could define addition and multiplication with these pictures more formally so we could explain in detail why things work out. However since the pictures are for motivation, and are actually difficult to write out on paper, it doesn't make much sense to work out these details. We will see an explanation in another context later on.

4.2.1 Picture functions

As you've seen, in our descriptions of ways of choosing fruits, we've treated the pictures of the fruit as if they are variables. You've also likely noticed that it is much easier to do algebraic manipulations with letters rather than pictures, simply because it is time consuming to draw the same picture over and over again, while we are used to writing letters quickly. In the theory of generating functions, we associate variables or polynomials or even power series with members of a set. There is no standard language describing how we associate variables with members of a set, so we shall invent some. By a *picture* of a member of a set we will mean a variable, or perhaps a product of powers of variables (or even a sum of products of powers of variables). A function that assigns a picture $P(s)$ to each member s of a set S will be called a *picture function*. The **picture enumerator** for a picture function P defined on a set S will be

$$E_P(S) = \sum_{s:s\in S} P(x).$$

We choose this language because the picture enumerator lists, or enumerates, all the elements of S according to their pictures. Thus Formula 4.5 is the picture enumerator the set of all multisets of fruit with between one and three apples, one and two pears, and one and two bananas.

164. How would you write down a polynomial in the variable A that says you should take between zero and three apples?

165. How would you write down a picture enumerator that says we take between zero and three apples, between zero and three pears, and between zero and three bananas?

166. Notice that if we use A^2 to stand for taking two apples, and P^3 to stand for taking three pears, then we have used the product A^2P^3 to stand for taking two apples and three pears. Thus we have chosen the picture of the ordered pair (2 apples,3 pears) to be the product of the pictures of a multiset of two apples and a multiset of three pears. Show that if S_1 and S_2 are sets with picture functions P_1 and P_2 defined on them, and if we define the picture of an ordered pair $(x_1, x_2) \in S_1 \times S_2$ to be $P((x_1, x_2)) = P_1(x_1)P_2(x_2)$, then the picture enumerator of P on the set $S_1 \times S_2$ is $E_{P_1}(S_1)E_{P_2}(S_2)$. We call this the **product principle for picture enumerators**.

4.2.2 Generating functions

167. Suppose you are going to choose a snack of between zero and three apples, between zero and three pears, and between zero and three bananas. Write down a polynomial in one variable x such that the coefficient of x^n is the number of ways to choose a snack with n pieces of fruit? Hint: substitute something for A, P and B in your formula from Problem 165.

168. Suppose an apple costs 20 cents, a banana costs 25 cents, and a pear costs 30 cents. What should you substitute for A, P, and B in Problem 165 in order to get a polynomial in which the coefficient of x^n is the number of ways to choose a selection of fruit that costs n cents?

169. Suppose an apple has 40 calories, a pear has 60 calories, and a banana has 80 calories. What should you substitute for A, P, and B in Problem 165 in order to get a polynomial in which the coefficient of x^n is the number of ways to choose a selection of fruit that has n calories?

170. We are going to choose a subset of the set $[n] = \{1, 2, \ldots, n\}$. Suppose we use x_1 to be the picture of choosing 1 to be in our subset. What is the picture enumerator for either choosing 1 or not choosing 1? Suppose that for each i between 1 and n, we use x_i to be the picture of choosing i to be in our subset. What is the picture enumerator for either choosing i or not choosing i to be in our subset? What is the picture enumerator for all possible choices of subsets of $[n]$? What should we substitute for x_i in order to get a polynomial in x such that the coefficient of x^k is the number of ways to choose a k-element subset of n? What theorem have we just reproved (a special case of)?

In Problem 170 we see that we can think of the process of expanding the polynomial $(1+x)^n$ as a way of "generating" the binomial coefficients $\binom{n}{k}$ as the coefficients of x^k in the expansion of $(1+x)^n$. For this reason, we say that $(1+x)^n$ is the "generating function" for the binomial coefficients $\binom{n}{k}$. More generally, the **generating function** for a sequence a_i, defined for i with $0 \le i \le n$ is the expression $\sum_{i=0}^{n} a_i x^i$, and the **generating function** for the sequence a_i with $i \ge 0$ is the expression $\sum_{i=0}^{\infty} a_i x^i$. This last expression is an example of a power series. In calculus it is important to think about whether a power series converges in order to determine whether or not it represents a function. In a nice twist of language, even though we use the phrase generating function as the name of a power series in combinatorics, we don't require the power series to actually represent a function in the usual sense, and so we don't have to worry about convergence.[3]

4.2.3 Power series

For now, most of our uses of power series will involve just simple algebra. Since we use power series in a different way in combinatorics than we do in calculus, we should review a bit of the algebra of power series.

171. In the polynomial $(a_0+a_1x+a_2x^2)(b_0+b_1x+b_2x^2+b_3x^3)$, what is the coefficient of x^2? What is the coefficient of x^4?

172. In Problem 171 why is there a b_0 and a b_1 in your expression for the coefficient of x^2 but there is not a b_0 or a b_1 in your expression for the coefficient of x^4? What is the coefficient of x^4 in

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4)?$$

Express this coefficient in the form

$$\sum_{i=0}^{4} \text{something},$$

where the something is an expression you need to figure out. Now suppose that $a_3 = 0$, $a_4 = 0$ and $b_4 = 0$. To what is your expression equal after you substitute these values? In particular, what does this have to do with Problem 171?

173. The point of the Problems 171 and 172 is that so long as we are willing to assume $a_i = 0$ for $i > n$ and $b_j = 0$ for $j > m$, then there is a very nice formula for the coefficient of x^k in the product

$$(\sum_{i=0}^{n} a_ix^i)(\sum_{j=0}^{m} b_jx^j).$$

Write down this formula explicitly.

[3] In the evolution of our current mathematical terminology, the word function evolved through several meanings, starting with very imprecise meanings and ending with our current rather precise meaning. The terminology "generating function" may be though of as an example of one of the earlier usages of the term function.

174. Assuming that the rules you use to do arithmetic with polynomials apply to power series, write down a formula for the coefficient of x^k in the product
$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j).$$

We use the expression you obtained in Problem 174 to *define* the product of power series. That is, we define the product

$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j)$$

to be the power series $\sum_{i=0}^{\infty} c_k x^k$, where c_k is the expression you found in Problem 174. Since you derived this expression by using the usual rules of algebra for polynomials, it should not be surprising that the product of power series satisfies these rules.[4]

4.2.4 Product principle for generating functions

Each time that we converted a picture function to a generating function by substituting x or some power of x for each picture, the coefficient of x had a meaning that was significant to us. For example, with the picture enumerator for selecting between zero and three each of apples, pears, and bananas, when we substituted x for each of our pictures, the exponent i in the power x^i is the number of pieces of fruit in the fruit selection that led us to x^i. After we simplify our product by collecting together all like powers of x, the coefficient of x^i is the number of fruit selections that use i pieces of fruit. In the same way, if we substitute x^c for a picture, where c is the number of calories in that particular kind of fruit, then the i in an x^i term in our generating function stands for the number of calories in a fruit selection that gave rise to x^i, and the coefficient of x^i in our generating function is the number of fruit selections with i calories. The product principle of picture enumerators translates directly into a product principle for generating functions.

175. Suppose that we have two sets S_1 and S_2. Let v_1 (v stands for value) be a function from S_1 to the nonnegative integers and let v_2 be a function

[4] Technically we should explicitly state these rules and prove that they are all valid for power series multiplication, but it seems like overkill at this point to do so!

from S_2 to the nonnegative integers. Define a new function v on the set $S_1 \times S_2$ by $v(x_1, x_2) = v_1(x_1) + v_2(x_2)$. Suppose further that $\sum_{i=0}^{\infty} a_i x^i$ is the generating function for the number of elements x_1 of S_1 of value i, that is with $v_1(x_1) = i$. Suppose also that $\sum_{j=0}^{\infty} b_j x^j$ is the generating function for the number of elements of x_2 of S_2 of value j, that is with $v_2(x_2) = j$. Prove that the coefficient of x^k in

$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j)$$

is the number of ordered pairs (x_1, x_2) in $S_1 \times S_2$ with total value k, that is with $v_1(x_1) + v_2(x_2) = k$. This is called the **product principle for generating functions**.

176. Let i denote an integer between 1 and n.

    (a) What is the generating function for the number of subsets of $\{i\}$ of each possible size? (Notice that the only subsets of $\{i\}$ are $\emptyset$ and $\{i\}$.)

    (b) Use the product principle for generating functions to prove the binomial theorem.

4.2.5 The extended binomial theorem and multisets

177. Suppose once again that i is an integer between 1 and n.

    (a) What is the generating function in which the coefficient of x^k is the number of multisets of size k chosen from $\{i\}$?

    (b) Express generating function in which the coefficient of x^k is the number of multisets chosen from $[n]$ as a power of a power series. What does Problem 108 (in which your answer could be expressed as a binomial coefficient) tell you about what this generating function equals?

178. What is the product $(1-x)\sum_{k=0}^{n} x^k$? What is the product

$$(1-x)\sum_{i=0}^{\infty} x^k?$$

179. Express the generating function for the number of multisets of size k chosen from $[n]$ (where n is fixed but k can be any nonnegative integer) as a 1 over something relatively simple.

180. Find a formula for $(1+x)^{-n}$ as a power series whose coefficients involve binomial coefficients. What does this formula tell you about how we should define $\binom{-n}{k}$ when n is positive?

181. If you define $\binom{-n}{k}$ in the way you described in Problem 180, you can write down a version of the binomial theorem for $(x+y)^n$ that is valid for both nonnegative and negative values of n. Do so. This is called the *extended binomial theorem*.

182. Write down the generating function for the number of ways to distribute identical pieces of candy to three children so that no child gets more than 4 pieces. Write this generating function as a quotient of polynomials. Using both the extended binomial theorem and the original binomial theorem, find out in how many ways we can pass out exactly ten pieces. Use one of our earlier counting techniques to verify your answer.

183. What is the generating function for the number of multisets chosen from an n-element set so that each element appears at least j times and less than m times. Write this generating function as a quotient of polynomials, then as a product of a polynomial and a power series.

4.2.6 Generating functions for integer partitions

184. If we have five identical pennies, five identical nickels, five identical dimes, and five identical quarters, give the picture enumerator for the combinations of coins we can form and convert it to a generating function for the number of ways to make k cents with the coins we have. Do the same thing assuming we have an unlimited supply of pennies, nickels, dimes, and quarters.

185. Recall that a partition of an integer n is a multiset of numbers that adds to n. In Problem 184 we found the generating function for the number of partitions of an integer into parts of size 1, 5, 10, and 25. Give the generating function for the number partitions of an integer into parts

of size one through ten. Give the generating function for the number of partitions of an integer into parts of any size. This last generating function involves an infinite product. Describe the kinds of terms you actually multiply and add together to get the last generating function. Rewrite any power series that appear in your product as quotients of polynomials or as integers divided by polynomials.

186. What is the generating function for the number of partitions of an integer in which each part is even?

187. What is the generating function for the number of partitions of an integer into distinct parts, that is, in which each part is used at most once?

188. Use generating functions to explain why the number of partitions of an integer in which each part is used an even number of times equals the generating function for the number of partitions of an integer in which each part is even.

189. Use the fact that

$$\frac{1-x^{2i}}{1-x^i} = 1+x^i$$

and the generating function for the number of partitions of an integer into distinct parts to show how the number of partitions of an integer n into distinct parts is related to the number of partitions of an integer n into odd parts.

190. Write down the generating function for the number of ways to partition an integer into parts of size no more than m, each used an odd number of times. Write down the generating function for the number of partitions of an integer into parts of size no more than m, each used an even number of times. Use these two generating functions to get a relationship between the two sequences for which you wrote down the generating functions.

4.3 Generating Functions and Recurrence Relations

Recall that a recurrence relation for a sequence a_n expresses a_n in terms of values a_i for $i < n$. For example, the equation $a_i = 3a_{i-1} + 2^i$ is a first order linear constant coefficient recurrence.

4.3.1 How generating functions are relevant

Algebraic manipulations with generating functions can sometimes reveal the solutions to a recurrence relation.

191. Suppose that $a_i = 3a_{i-1} + 2^i$.

    (a) Multiply both sides by x^i and sum both the left hand side and right hand side from $i = 1$ to infinity. In the left-hand side use the fact that
    $$\sum_{i=1}^{\infty} a_i x^i = (\sum_{i=0}^{\infty} x^i) - a_0$$
    and in the right hand side, use the fact that
    $$\sum_{i=1}^{\infty} a_{i-1} x^i = x \sum_{i=0}^{\infty} a_i x^i$$
    to rewrite the equation in terms of the power series $\sum_{i=0}^{\infty} a_i x^i$. Solve the resulting equation for the power series $\sum_{i=0}^{\infty} a_i x^i$

    (b) Use the previous part to get a formula for a_i in terms of a_0.

192. Suppose we deposit \$5000 in a savings certificate that pays ten percent interest and also participate in a program to add \$1000 to the certificate at the end of each year that follows (also subject to interest.) Assuming we make the \$5000 deposit at the end of year 0, and letting a_i be the amount of money in the account at the end of year i, write a recurrence for the amount of money the certificate is worth at the end of year n. Solve this recurrence. How much money do we have in the account at the end of ten years? At the end of 20 years?

4.3.2 Fibonacci Numbers

The sequence of problems that follows describes a number of hypotheses we might make about a fictional population of rabbits. We use the example of a rabbit population for historic reasons; our goal is a classical sequence of numbers called Fibonacci numbers. When Fibonacci introduced them, he did so with a fictional population of rabbits.

4.3.3 Second order linear recurrence relations

193. Suppose we start with 10 pairs of baby rabbits, and that after baby rabbits mature for one month they begin to reproduce, with each pair producing two new pairs at the end of each month afterwards. Suppose further that over the time we observe the rabbits, none die. Show that $a_n = a_{n-1} + 2a_{n-2}$. This is an example of a *second order linear* recurrence with constant coefficients. Using the method of Problem 191, show that
$$\sum_{i=0}^{\infty} a_i x^i = \frac{10}{1 - x - 2x^2}.$$
This gives us the generating function for the sequence a_i giving the population in month i; shortly we shall see a method for converting this to a solution to the recurrence.

194. In Fibonacci's original problem, each pair of mature rabbits produces one new pair at the end of each month, but otherwise the situation is the same as in Problem 193. Assuming that we start with one pair of baby rabbits (at the end of month 0), find the generating function for the number of pairs of rabbits we have at the end on n months.

195. Find the generating function for the solutions to the recurrence
$$a_i = 5a_{i-1} - 6a_{i-2} + 2^i.$$

The recurrence relations we have seen in this section are called *second order* because they specify a_i in terms of a_{i-1} and a_{i-2}, they are called *linear* because a_{i-1} and a_{i-2} each appear to the first power, and they are called *constant coefficient recurrences* because the coefficients in front of a_{i-1} and a_{i-2} are constants.

4.3.4 Partial fractions

The generating functions you found in the previous section all can be expressed in terms of the reciprocal of a quadratic polynomial. However without a power series representation, the generating function doesn't tell us what the sequence is. It turns out that whenever you can factor a polynomial into linear factors (and over the complex numbers such a factorization always exists) you can use that factorization to express the reciprocal in terms of power series.

196. Express $\frac{1}{x-3} + \frac{2}{x-2}$ as a single fraction.

197. In Problem 196 you see that when we added numerical multiples of the reciprocals of first degree polynomials we got a fraction in which the denominator is a quadratic polynomial. This will always happen unless the two denominators are multiples of each other, because their least common multiple will simply be their product, a quadratic polynomial. This leads us to ask whether a fraction whose denominator is a quadratic polynomial can always be expressed as a sum of fractions whose denominators are first degree polynomials. Find numbers a and b so that
$$\frac{5x+1}{(x-3)(x+5)} = \frac{c}{x-3} + \frac{d}{x-5}.$$

198. In Problem 197 you may have simply guessed at values of c and d, or you may have solved a system of equations in the two unknowns c and c. Given constants a, b, r_1, and r_2 (with $r_1 \neq r_2$, write down a system of equations we can solve for c and d to write
$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}.$$

Writing down the equations in Problem 198 and solving them is called the *method of partial fractions.* This method will let you find power series expansions for generating functions of the type you found in Problems 193 to 195. However you have to be able to factor the quadratic polynomials that are in the denominators of your generating functions.

199. Use the quadratic formula to find the solutions to $x^2 - x - 1 = 0$, and use that information to factor $x^2 - x - 1$.

200. Use the factors you found in Problem 199 to write
$$\frac{1}{x^2 - x - 1}$$
in the form
$$\frac{1}{x - r_1} + \frac{1}{x - r_2}.$$
Hint: You can save yourself a tremendous amount of frustrating algebra if you arbitrarily choose one of the solutions and call it r_1 and call the other solution r_2 and solve the problem using these algebraic symbols in place of the actual roots.[5] Not only will you save yourself some work, but you will get a formula you could use in other problems. When you are done, substitute in the actual values of the solutions and simplify.

201. Use the partial fractions decomposition you found in Problem 199 to write the generating function you found in Problem 194 in the form
$$\sum_{i=0}^{\infty} a_i x^i$$
and use this to give an explicit formula for a_n. (Hint: once again it will save a lot of tedious algebra if you use the symbols r_1 and r_2 for the solutions as in Problem 200 and substitute the actual values of the solutions once you have a formula for a_n in terms of r_1 and r_2.) When we have $a_0 = 1$ and $a_1 = 1$, i.e. when we start with one pair of baby rabbits, the numbers a_n are called Fibonacci Numbers. Use either the recurrence or your final formula to find a_2 through a_8. Are you amazed that your general formula produces integers, or for that matter produces rational numbers? Why does the recurrence equation tell you that the Fibonacci numbers are all integers? Try to find an algebraic explanation (not using the recurrence equation) of why the formula has to do so. Explain why there is a real number b such that the value of the nth Fibonacci number is almost exactly (but not quite) some constant times b^n. (Find b and the constant.)

202. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$.

[5] We use the words roots and solutions interchangeably.

4.3.5 Catalan Numbers

203. Using either lattice paths or diagonal lattice paths, explain why the Catalan Number c_n satisfies the recurrence

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i}.$$

Show that if we use y to stand for the power series $\sum_{i=0}^{\infty} c_n x^n$, then we can find y by solving a quadratic equation. Solve for y. Taylor's theorem from calculus tells us that the extended binomial theorem

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$$

holds for any number real number r, where $\binom{r}{i}$ is defined to be

$$\frac{r^{\underline{i}}}{i!} = \frac{r(r-1)\cdots(r-i+1)}{i!}.$$

Use this and your solution for y to get a formula for the Catalan numbers.

4.4 Supplementary Problems

1. Each person attending a party has been asked to bring a prize. The person planning the party has arranged to give out exactly as many prizes as there are guests, but any person may win any number of prizes. If there are n guests, in how many ways may the prizes be given out so that nobody gets the prize that he or she brought?

2. There are m students attending a seminar in a room with n seats. The seminar is a long one, and in the middle the group takes a break. In how many ways may the students return to the room and sit down so that nobody is in the same seat as before?

3. In how many ways may k distinct books be arranged on n shelves so that no shelf gets more than m books?

4. A group of n married couples comes to a group discussion session where they all sit around a round table. In how many ways can they sit so that no person is next to his or her spouse?

5. A group of n married couples comes to a group discussion session where they all sit around a round table. In how many ways can they sit so that no person is next to his or her spouse or a person of the same sex? This problem is called the *menage problem*. (Hint: Reason as you did in Problem 4, noting that if the set of couples who do sit side-by-side is nonempty, then the sex of the person at each place at the table is determined once we seat one couple in that set.)

6. What is the generating function for the number of ways to pass out k pieces of candy from an unlimited supply of identical candy to n children (where n is fixed) so that each child gets between three and six pieces of candy (inclusive)? Use the fact that $(1 + x + x + x^3)(1 - x) = 1 - x^4$ to find a formula for the number of ways to pass out the candy. Reformulate this problem as an inclusion-exclusion problem and describe what you would need to do to solve it.

7. Find a recurrence relation for the number of ways to divide a convex n-gon into triangles by means of non-intersecting diagonals. How do these numbers relate to the Catalan numbers?

8. How does $\sum_{k=0}^{n} \binom{n-k}{k}$ relate to the Fibonacci Numbers?

9. Let m and n be fixed. Express the generating function for the number of k-element multisets of an n-element set such that no element appears more than m times as a quotient of two polynomials. Use this expression to get a formula for the number of k-element multisets of an n-element set such that no element appears more than m times.

10. One natural but oversimplified model for the growth of a tree is that all new wood grows from the previous year's growth and is proportional to it in amount. To be more precise, assume that the (total) length of the new growth in a given year is the constant c times the (total) length of new growth in the previous year. Write down a recurrence for the total length a_n of all the branches of the tree at the end of growing season n. Find the general solution to your recurrence relation. Assume that we begin with a one meter cutting of new wood which branches out and

grows a total of two meters of new wood in the first year. What will the total length of all the branches of the tree be at the end of n years?