# Selected Answers to 3.4 and 4.4 Suggested Problems

## 3.4

10. (a) $S$ is clearly linearly independent, so we need only verify it is in $V$. $0-2+3-1+0=0$, so it is.

(b) We need a spanning set for $V$ to which we can add $S$, then pare down into a basis. $V$ has dimension 4, since determining any four entries fixes the fifth, but determining 3 leaves you freedom. Some vectors of $V$: $(1,0,0,1,0)$, $(0,1,0,0,1)$, $(1,0,-1,0,1)$, $(2,1,0,0,0)$. This is a linearly independent set and hence a basis. If we make a matrix with $S$ as the first column (this ensures $S$ will be included in our basis) and this basis as the remaining four columns and row-reduce it, we get

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the first, second, third, and fifth columns correspond to the first four elements of the standard basis, they correspond to our basis elements. Hence, a basis for $V$ which includes $S$ would be

$$\{(0,1,1,1,0),(1,0,0,1,0),(0,1,0,0,1),(2,1,0,0,0)\}$$

(among many others).

14. We will prove by contrapositive. Suppose that $A$ is not in reduced row echelon form. Then at least one of the following applies:
(a) A row of all zeros is above a row with a nonzero entry.
(b) The first nonzero entry in some row is not the only nonzero entry in its column.
(c) The first nonzero entry in some row is $c \neq 1$.
(d) The first nonzero entry in some row is to the left of the first nonzero entry in a higher row.

It is clear that if (b), (c), or (d) applies, $(A|b)$ cannot be in reduced row echelon form, because the addition of $b$ will not affect the first nonzero entry in a row of $A$. Hence we must only worry about (a).

If the entry of $b$ corresponding to the zero row of $A$ in question is zero, then $(A|b)$ is not in reduced row echelon form because it has a row of all zeros above a row which is not all zero. If the entry of $b$ is nonzero, then $(A|b)$ has property (d) above, since the nonzero row of $A$ below the all-zero row of $A$ is now a row whose first nonzero entry is to the left of the first nonzero entry (the entry of $b$) of a higher row. Hence $A$ not in reduced row echelon form implies $(A|b)$ is not in reduced row echelon form.

## 4.4

5. There are two ways to approach this. One is to say that $M$ can be put into upper triangular form by dealing only with rows of $(A\ B)$, and then the determinant is taken by the product of the diagonal elements. That product will be the product of the diagonal elements of the altered $A$ and will have the same relationship to the determinant of $M$ as it

does to the determinant of $A$ (in the sense of sign; we may have had to perform some odd number of row swaps to get it to upper triangular form), and hence $\det(M) = \det(A)$.

The other method is by successive cofactor expansion along the last row of $M$ and its submatrices. Since $A$ and $I$ are square, $M$ must also be square. Suppose $A$ is $k \times k$ and $M$ is $n \times n$ for some $n \geq k$. Let $M_i$ denote the $i \times i$ matrix taken from the first $i$ rows and columns of $M$, so $M_n = M$ and $M_k = A$. Then by cofactor expansion, since all but one entry of each of the last $n-k$ rows of $M$ is zero,

$$\det(M) = 1 \cdot \det(M_{n-1}) = 1 \cdot 1 \cdot \det(M_{n-2}) = \ldots = 1 \cdot \ldots \cdot 1 \cdot \det(M_k) = \det(A).$$

6. It is easy to show that

$$\begin{pmatrix} I & B \\ 0 & C \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

since $A$ and $C$ are square. By exercise 5 and the fact that $\det(XY) = \det(X)\det(Y)$, we get that the determinant of $M$ is $\det(A)\det(C)$.