Selected answers to assignment 7: 2.5, 3.1

2.5

9. "is similar to" is an equivalence relation:
reflexive: $A = I^{-1}AI$
symmetric: if $B = Q^{-1}AQ$, then $A = (Q^{-1})^{-1}B(Q^{-1})$.
transitive: if $B = Q^{-1}AQ$ and $C = P^{-1}BP$, then $C = P^{-1}Q^{-1}AQP = (QP)^{-1}A(QP)$.

12. The part of the corollary requiring proof is that the equality holds. Q is the change of coordinate matrix from γ coordinates to β (= standard basis) coordinates automatically (see the top of page 112). The equality's proof comes from remembering L_A is defined specifically with respect to the standard basis β and applying Theorem 2.23.

13. If β' is a basis, then definitionally Q is the change-of-coordinate matrix. Hence we need only prove β' is a basis. Suppose there is a nontrivial representation of zero from elements of β'; in other words,

$$a_1x'_1 + a_2x'_2 + \ldots + a_nx'_n = 0$$

for some a_i not all zero. Then by definition of β', we get

$$a_1\sum_{i=1}^{n} Q_{i1}x_i + a_2\sum_{i=1}^{n} Q_{i2}x_i + \ldots + a_n\sum_{i=1}^{n} Q_{in}x_i = 0.$$

Rearrange the sums and collect together the different x_i:

$$\left(\sum_{j=1}^{n} a_jQ_{1j}\right)x_1 + \left(\sum_{j=1}^{n} a_jQ_{2j}\right)x_2 + \ldots + \left(\sum_{j=1}^{n} a_jQ_{nj}\right)x_n = 0.$$

This is a representation of 0 by vectors from β and hence must be trivial, so for all i,

$$\sum_{j=1}^{n} a_jQ_{ij} = 0$$

In other words, there is a linear combination of the columns of Q which gives zero. Since Q is invertible, it must be a trivial representation of zero, and finally we see $a_i = 0$ for all i (phew!).

3.1

3. (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. The key is that elementary matrices always are obtained by applying a single elementary operation to the identity matrix, and therefore we can characterize them well. If E is our elementary matrix and it is obtained by swapping the i, jth rows, then E has entries which are identical to the identity in rows other than i, and j, and in those rows it has a single 1: entry ij and ji (instead of ii and jj. However, this is exactly what is accomplished by swapping columns i and j.

Likewise, multiplying row i by c is equivalent to multiplying column i by c.

For the third sort of operation we must be a little more careful. Suppose E is obtained by multiplying the ith row of I by c and adding it to the jth row. Then the only difference between E and I is a c in the jith position. This can be equivalently gotten by multiplying the jth column of I and adding it to the ith row (note the interchange of indices).

6. This should be clear. Probably the quickest way to prove it is to say that if E is elementary and $EA = B$, then $A^t E^t = B^t$, and E on the left and E^t on the right are corresponding row and column operations.