Selected answers to assignment 6: 2.4

2. (a), (b), (d), (e) noninvertible; dimensions wrong.

(c) invertible; standard basis mapped to $(3,0,3), (0,1,4), (-2,0,0)$, which is a basis.

(f) invertible; standard basis mapped to $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which is a basis.

3. (b), (c) are isomorphic pairs; (a), (d) are not. All by dimension.

4. Multiply AB by $B^{-1}A^{-1}$; replace the inverse pairs by I_n from the inside-out.

5. The transpose of a product is the product of the transposes in reverse order. Apply that to AA^{-1} or the reverse.

6. Since A is invertible there is some A^{-1}. Then $A^{-1}AB = A^{-1}0$, which simplifies to $B = 0$.

10. (a) Since I_n is invertible and A, B are square, exercise 9 applies and shows A and B are invertible.

(b) Since A and B are invertible we may multiply the equality $AB = I_n$ on the left by A^{-1} or on the right by B^{-1}; simplification gives $B = A^{-1}$ and $A = B^{-1}$.

(c) If V and W are n-dimensional vector spaces and $T : V \to W$, $U : W \to V$ are such that $UT = I_V$, then T and U are invertible and are in fact each others' inverses. Proof by applying (a) and (b) to $[UT]_\beta$.

13. Isomorphism is reflexive: $V \sim V$ as witnessed by I_V.

Symmetric: if $V \sim W$ is witnessed by T, then T^{-1} witnesses $W \sim V$.

Transitive: if $V \sim W$ and $W \sim Z$, shown by T and U respectively, then $V \sim Z$ is shown by UT.

16. $B^{-1}(cA + D)B = cB^{-1}AB + B^{-1}DB$ by a few applications of Theorem 2.12, p. 89. Use exercise 6 twice to argue that if $B^{-1}AB = 0$, A must be 0. Hence the null space is zero and Φ is one-to-one. Since the vector space is finite-dimensional that suffices to show Φ is also onto and hence an isomorphism.

20. Since ϕ_β is an isomorphism, $R(L_A\phi_\beta) = R(L_A)$. Since ϕ_γ is an isomorphism, by #17 $R(\phi_\gamma T)$ has equal rank to $R(T)$. Commutativity of Figure 2.2 then shows $\text{rank}(T) = \text{rank}(L_A)$.

The nullity argument is similar.