Selected answers to assignment 5, 2.2–2.3

2.2

5. (c) (1 0 0 1)
(d) (1 2 4)
(f) $\begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$
(g) (a)

8. If x is a linear combination of basis vectors with coefficients a_i and y with coefficients b_i, then

$$T(cx+y) = \begin{pmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = cT(x) + T(y).$$

11. Let α be a basis for W and extend it to a basis β for all of V, ordered so that the vectors from α are listed first. Since $T(W) \subseteq W$, all elements of α will be mapped by T to linear combinations of elements of α. That is, their image coordinate vectors will have zeros in the $(k+1)^{st}$ through n^{th} places. Since the images of the basis vectors form the columns of $[T]_\beta$, the first k columns will be zero from row $k+1$ down, and $[T]_\beta$ has the required form.

13. We want to show that if $aT + bU = T_0$, we must have $a = b = 0$. We know that if one of a or b is nonzero, the other must also be, because neither T nor U is T_0 and hence neither is a multiple of T_0. If

$$(aT + bU)(x) = \mathit{0}, \quad \text{then} \quad aT(x) + bU(x) = \mathit{0}$$

by definition. Hence $aT(x) = -bU(x)$, so we must have $T(ax) = U(-bx)$. Both these values are in the range of their respective linear transformation, so both must be zero. If a and b are nonzero, x must be in $N(T) \cap N(U)$ to get this equality. However, since neither T nor U is T_0, there are elements y outside $N(T) \cap N(U)$, and on such y $aT + bU$ will give nonzero output. Therefore $aT + bU = T_0$ only when $a = b = 0$, and $\{U, T\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

15. (a) Show S^0 is a subspace of $\mathcal{L}(V, W)$:
$T_0 \in S^0$ because $T_0(x) = \mathit{0}$ for all x, including those in S. If $T, U \in S^0$, $(aT + U)(x) = aT(x) + U(x)$ which is zero for all $x \in S$, so $aT + U \in S^0$.
(b) Show that $S_1 \subseteq S_2 \Rightarrow S_2^0 \subseteq S_1^0$:
Suppose $T \in S_2^0$. Then $T(x) = \mathit{0}$ for all $x \in S_2$. Since $S_1 \subseteq S_2$, this includes all $x \in S_1$, so $T \in S_1^0$.
(c) Show that $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$:
$\subseteq$: Suppose $T(v_1 + v_2) = \mathit{0}$ for all $v_1 \in V_1$, $v_2 \in V_2$. Then in particular, $T(v_1 + \mathit{0}) = \mathit{0}$ and $T(\mathit{0} + v_2) = \mathit{0}$, so $T \in V_1^0$ and $T \in V_2^0$.
$\supseteq$: Suppose $T \in V_1^0 \cap V_2^0$. Then for all $v_i \in V_i$, $T(v_1 + v_2) = T(v_1) + T(v_2) = \mathit{0} + \mathit{0} = \mathit{0}$.

2.3

4. (b) $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$

(d) (12)

11. Suppose $T^2 = T_0$. Then $T(T(x)) = \mathit{0}$ for all x, so $T(y) = \mathit{0}$ for all $y \in R(T)$. Hence $R(T) \subseteq N(T)$. Now suppose $R(T) \subseteq N(T)$. $T(T(x)) = T(y)$ for some $y \in R(T)$, so $T(y) = \mathit{0}$. Hence for all x $T^2(x) = \mathit{0}$ and $T^2 = T_0$.

12. (a) Show UT one-to-one implies T one-to-one:

Suppose some $x_1 \neq x_2 \in V$ are such that $T(x_1) = T(x_2)$. Then $U(T(x_1)) = U(T(x_2))$ so $(UT)(x_1) = (UT)(x_2)$. Hence if T is not one-to-one, neither can UT be, so if UT is one-to-one, T must be also.

However, U need not be one-to-one on all of W, just on $R(T)$. E.g., let $T : \mathbb{R} \to \mathbb{R}^2$ be $T(a) = (a, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}$ be $U(a, b) = a$.

(b) Show UT onto implies U onto:

Suppose UT is onto, so for any $y \in Z$ there is some $x \in V$ such that $UT(x) = y$. But then $U(T(x)) = y$ for some $T(x) \in W$, so U is onto.

Same example as in (a) shows that T need not be onto.

(c) Show U, T one-to-one and onto implies UT one-to-one and onto:

Let $x_1 \neq x_2 \in V$. Then $T(x_1) \neq T(x_2)$ since T is one-to-one, and hence $U(T(x_1)) \neq U(T(x_2))$ since U is one-to-one. Therefore UT is one-to-one.

Now suppose $z \in Z$. Since U is onto there is $w \in W$ so that $U(w) = z$, and since T is onto there is $v \in V$ such that $T(v) = w$. Therefore $UT(v) = z$ and UT is onto.

15. The j^{th} column of MA is

$$\begin{pmatrix} \sum_{k=1}^{n} M_{1k}A_{kj} \\ \vdots \\ \sum_{k=1}^{n} M_{mk}A_{kj} \end{pmatrix}$$

and we may write

$$A_{kj} = \sum_{i=1}^{r} a_{\ell_i} A_{k\ell_i}$$

for all $1 \leq k \leq n$. Rearrange the resulting formula for M's tj^{th} entry into

$$\sum_{i=1}^{r} a_{\ell_i} \sum_{k=1}^{n} M_{tk} A_{k\ell_i}$$

and note this is exactly the linear combination corresponding to the one in A, as required.

17. Projections are the answer (see exercises from §2.1, especially #26).

projection $\Rightarrow T = T^2$: For any sum, $T(w_1 + w_2) = w_1 = w_1 + \mathit{0}$, so $T^2(w_1 + w_2) = T(w_1 + \mathit{0}) = w_1 = T(w_1 + w_2)$.

$T = T^2 \Rightarrow$ projection: Certainly $\{y : T(y) = y\} \cap N(T) = \{\mathit{0}\}$. For any $x \in V$ we may write $x = x - T(x) + T(x)$. By assumption, $x - T(x) \in N(T)$, since $T(x - T(x)) =$

$T(x) - T^2(x)$. Also by assumption, $T(x) \in \{y : T(y) = y\}$, since $T(T(x)) = T^2(x) = T(x)$. Thus as long as $\{y : T(y) = y\}$ is a subspace of V, we get $V = \{y : T(y) = y\} \oplus N(T)$ and T is a projection.

For subspace, let $x, y \in \{y : T(y) = y\}$. $T(cx + y) = cT(x) + T(y) = cx + y$ for $cx + y \in \{y : T(y) = y\}$.