Selected Answers to 1.6 and 2.1 Assignments

Math 24, Spring 2006

Section 1.6

2(a,b): (a) yes, (b) no

3(a,b): (a) no, (b) yes

5: no, it is too big

11: a matter of showing linear independence. For $\{u_v, au\}$, if you set up the equation $c(u+v) + d(au) = \mathit{0}$, after rearranging you get that since u, v are linearly independent, the values $c + da$ and c must be zero. Since c is zero and a is nonzero, that means d must also be zero and the only representation of $\mathit{0}$ is the trivial one. $\{au, bv\}$ is even more straightforward, still using the fact that both a and b are nonzero.

12: This is like 11. If you call the coefficients of the linear combination a, b, c, you get $c = 0$ which implies $b = 0$ which together imply $a = 0$.

14: For W_1, a_2 and a_5 are independent of the rest of the values, so $(0,1,0,0,0), (0,0,0,0,1)$ are basis vectors. For 1, 3, 4, we have to account for the interaction: if two are assigned values the third one is as well. In particular, if a_1 is 0, then a_3, a_4 are either $1, -1$ or $-1, 1$ (or some multiple thereof). We need only one of those: $(0,0,1,-1,0)$. If $a_1 = 1$, then $a_4 = 1 - a_3$, giving us vectors such as $(1,0,1,0,0)$, $(1,0,0,1,0)$, $(1,0,-1,2,0)$, etc. Any one of those is an acceptable choice, though, because combined with $(0,0,1,-1,0)$ it gives all the rest. Thus $\{(0,1,0,0,0), (0,0,0,0,1), (0,0,1,-1,0), (1,0,1,0,0)\}$ is a basis and $\dim(W_1) = 4$. That should make sense because instead of being able to deal with all five entries independently, fixing four of them (as long as that four includes both a_2 and a_5) fixes the fifth as well.

For W_2 basis elements are $(0,1,1,1,0)$ and $(1,0,0,0,-1)$ and the dimension is 2. Again, should make sense because if you pick the right two elements, fixing two elements fixes all the elements.

15: From the standard basis for $M_{n\times n}(F)$ we may keep the elements which have all zeros on the diagonal, since they have trace equal to zero. The standard basis has n^2 elements and we remove the n of them which have nonzero diagonal entries, leaving $n^2 - n$.

For the rest of the basis, consider the matrices which are all zero except for two entries: a 1 in the top left corner and a -1 somewhere else along the diagonal. They have trace zero and there are $n - 1$ of them; additionally they are linearly independent of each other and the basis elements above. We claim these generate all the diagonal zero trace matrices and hence with the basis elements above are a basis for all the zero trace matrices.

It should be clear that if we took every possible pair of diagonal entries and assigned 1 to one and -1 to the other, that set of matrices generates all the diagonal zero trace matrices, so we will simply show our set generates this set. Suppose M has a 1 in the i^{th} diagonal entry and a -1 in the j^{th} diagonal entry. If either of i, j is 1 we are finished, so suppose neither is. Let M_i be the matrix with entry $1,1$ equal to 1 and entry i,i equal to -1. Then $M = M_j - M_i$.

Therefore the zero trace matrices have a basis of size $(n^2 - n) + (n - 1) = n^2 - 1$.

22: Certainly it is sufficient that $W_1 \subseteq W_2$, since then $W_1 \cap W_2 = W_1$. It is also necessary, because $W_1 \cap W_2$ is always a subspace of W_1, so if the dimensions match it must be equal. The only way to have $W_1 \cap W_2 = W_1$ is to have $W_1 \subseteq W_2$.

Section 2.1

4: Null space is matrices of the form $\begin{pmatrix} \frac{1}{2}a_{12} & a_{12} & -2a_{12} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, dimension 4. Range is matrices with zero second row, dimension 2. Neither 1-1 nor onto.

5: Null space $\{\mathit{0}\}$ (image of ax^2+bx+c is $ax^3+bx^2+(2a+c)x+b$), range all polynomials where x^2's coefficient is the same as the constant term, dimension 3. 1-1 but not onto.

9: None of these distribute over sum, for various reasons. Additionally, (c) and (e) do not commute with scalar multiplication.

10: $T(2,3) = T(3(1,1)-(1,0)) = 3T(1,1) - T(1,0) = (6,15)-(1,4) = (5,11)$. T is 1-1 because the basis $(1,0),(1,1)$ is carried to a basis, and so $R(T) = \mathbb{R}^2$; from there the argument is either the dimension theorem or the consequence of the dimension theorem that between spaces of the same finite dimension, being 1-1 is equivalent to being onto.

14: (a) Suppose T is 1-1 and $T(L)$ is linearly dependent. Since T is 1-1, there are as many distinct vectors in $T(L)$ as in L. Since $T(L)$ is linearly dependent, there is a selection of vectors from $T(L)$, say $T(\ell_1), T(\ell_2), \ldots T(\ell_n)$, such that with scalars a_i not all zero, $\sum_{i=1}^n a_i T(\ell_i) = \mathit{0}$. By linearity, $T\left(\sum_{i=1}^n a_i \ell_i\right) = \mathit{0}$, and there is no cancellation of a_1 because the ℓ_i are all distinct vectors. By the fact that T is 1-1 and hence has zero kernel, $\sum_{i=1}^n a_i \ell_i = \mathit{0}$, a_i not all zero, and L is linearly dependent.

For the converse, note that for any nonzero vectore x, $\{x\}$ is a linearly independent set, but $\{\mathit{0}\}$ is linearly dependent. Therefore if T carries linearly independent sets to linearly independent sets, the only thing it can map to $\mathit{0}$ is $\mathit{0}$, and hence its kernel is $\{\mathit{0}\}$ and it is 1-1.

(b) By part (a), is S is linearly independent $T(S)$ must also be. Therefore suppose S is linearly dependent, and consider the image of $a_1s_1 + a_2s_2 + \ldots + a_ns_n = \mathit{0}$, a nontrivial representation of $\mathit{0}$ by vectors from S. On the one hand, the image must be $\mathit{0}$, and on the other is it $a_1T(s_1) + a_2T(s_2) + \ldots + a_nT(s_n)$, a linear combination of n distinct vectors from $T(S)$ (still n distinct vectors by the fact that T is 1-1), where not all a_i are zero (we have gotten no cancellation). Therefore $T(S)$ has a nontrivial representation of $\mathit{0}$ and is linearly dependent.

(c) By part (a) and T being 1-1, we know $T(\beta)$ is linearly independent, and by a previous theorem we know it spans $R(T)$. Since T is onto, $R(T) = W$, and $T(\beta)$ is thus a linearly independent spanning set for W, a.k.a. a basis.

15: Linearity follows from the rules that you can factor constants out of integrals and split integrals of sums into sums of integrals. By using the definite integral from 0 to x we have eliminated the $+C$ that comes along with indefinite integration, so each polynomial has a unique image (hence T is well-defined); therefore T is not onto, as it misses any polynomial with a nonzero constant term. Since derivatives are unique and the preimage under T is the derivative of the given polynomial, T is 1-1.

16: Every polynomial has a polynomial integral, so T is onto. However, every polynomial has an infinite family of integrals which vary from each other by constants, so T is not 1-1: if $f(x) = g(x) + k$, k a scalar, then $T(f(x)) = T(g(x))$.

17: Both parts of this basically boil down to having enough linearly independent vectors to work with. For (a), remember the image of a basis spans the range. That image has at most $\dim(V)$ linearly independent vectors, which is not enough to span all of W when $\dim(W) > \dim(V)$. For the latter, the image of a set of size $\dim(V)$ under T will necessarily be linearly dependent even if the original set was independent, because there isn't a size-$\dim(V)$ set of linearly independent vectors in W: they cut off at $\dim(W) < \dim(V)$.

20: Images and preimages of subspaces are subspaces. Clearly they both contain $\mathit{0}$. For the rest it is a matter of showing that linear combinations of elements of $T(V_1)$ are images of elements of V_1, and linear combinations of elements of $T^{-1}(W_1)$ (to abuse notation; there need not be an actual inverse to T) map to W_1 under T. Both use the simple properties of linearity.

26: Mostly these are a matter of writing down definitions - none of the proofs are long. For (c) consider the statement about W_1 in part (a): if W_1 is all of V, then T is the identity transformation. For (d) look at $R(T)$: if W_1 is the zero subspace, then T is the zero transformation.

28: $T(\mathit{0}) = \mathit{0}$ for any linear T; $T : V \to V$ so $T(V) \subseteq V$; $R(T) = T(V)$ so all images fall into it, including those of its own elements; $T(N(T)) = \{\mathit{0}\} \subseteq N(T)$ by definition of $N(T)$.

29: We know that T distributes over addition and scalar multiplication, and that it takes $\mathit{0}$ to $\mathit{0}$. Since W is T-invariant, T_W is a map from W to W. You can show easily, by appealing to $T_W = T$ wherever T_W is defined, that T_W distributes over addition and scalar multiplication and takes $\mathit{0}$ to $\mathit{0}$, using the fact that if either $ax + y$ or the pair x, y are in W, then the other is as well.