Mathematics 24

Winter 2004

Exam I, January 27, 2004

In-class Portion

YOUR NAME:

1. Complete the following definitions:

   (a) Suppose that V is a vector space over a field F. Then W is a *subspace* of V if:

   (b) Suppose that V is a vector space over a field F and $S \subseteq V$. The set S is *linearly independent* if:

   (c) Suppose that V and W are vector spaces over the same field F. A function $T : V \to W$ is *linear* if:

   (d) Suppose that $T : V \to W$ is a linear transformation. The *null space* of $T$is:

2. Identify each statement as true (**T**) or false (**F**.)

   (a) The union of subspaces of a vector space is always a subspace.

   (b) If L is a linearly independent subset of a vector space V, then L is a basis for $span(L)$.

   (c) Let G be a finite subset of a vector space V. Then G generates V if and only if some subset of G is a basis for V.

   (d) The following constitutes a complete description of a vector space:
   $V = \{(x, y) \mid x \in \mathbb{C} \ \& \ y \in \mathbb{C}\}$,
   $(x, y) + (z, w) = (x + z, y + w)$,
   $a(x, y) = (ax, ay)$.

   (e) A system of two linear equations in three variables of the form
   $$a_1x + b_1y + c_1z = 0$$
   $$a_2x + b_2y + c_2z = 0$$
   must have infinitely many solutions.

3. Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ and $\gamma = \{(1,0),\ (0,1)\}$ be ordered bases for $M_{2\times 2}(F)$ and F^2 respectively. Define linear functions T and U by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (a+d,\ b+c) \qquad\qquad U(\,(x,y)\,) = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

(a) Find:

$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]_\beta =$

$[(1,1)]_\gamma =$

(b) Find:

$[T]_\beta^\gamma =$

$[U]_\gamma^\beta =$

(c) Compute $[U(1,1)]_\beta$ as a product of items you found in part (a) and/or (b):

(d) Compute $[TU]_\gamma^\gamma$ as a product of items you found in part (a) and/or (b):

(e) What does your answer to (d) tell you about TU?

4. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear.

(a) Describe $N(T)$ geometrically, given the following geometric descriptions of $R(T)$:

i. $R(T)$ is all of $\mathbb{R}^3$.

ii. $R(T)$ is a plane.

iii. $R(T)$ is a line.

iv. $R(T)$ is the origin.

(b) Name and/or state a theorem that can be used to show your answers to (a) are correct.

(c) If T is the projection onto the xy-plane along the line $x = y = z$:

i. What is $N(T)$?

ii. What is $R(T)$?

Remember that if V is the direct sum of W_1 and W_2, the projection onto W_1 along W_2 is the function T computed as follows: Write $v \in V$ as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. Then $T(v) = w_1$.

(Scratch paper if needed.)

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Winter 2004
Exam I, January 27, 2004
Take-home Portion due 5 PM, Friday, January 30, 2004 **Corrected Version**

Be sure to include the statements of the problems in your solutions. Completed exams may be given to me in class on Friday, or brought to my office (104 Choate House) before 5 PM on Friday. (If I am not there, slide your paper under my door — preferably in an envelope.)

Please note: When working on this exam you may consult your textbook for this course, your class notes, and me (Prof. Groszek.) Please do not consult any other books or sources, or discuss these problems with anyone other than me.

If you are at all unsure what any of these questions means, or what is expected in the answer, please ask me, either in person or by e-mail.

1. Suppose that
$$W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots$$
is an infinite chain of subspaces of a vector space V. Show that the union
$$W = \bigcup_{n \in \mathbb{N}} W_n$$
is also a subspace of V.

   Note 1: Another way to write the union is
$$W = \{v \in V \mid (\exists n \in \mathbb{N})\ [v \in W_n]\}.$$

   Note 2: You may want to use the fact that if $x \in W_n$ and $y \in W_m$, then both x and y are in $W_{max(m,n)}$.

2. Suppose that L_1 and L_2 are disjoint linearly independent subsets of a vector space V, $W_1 = span(L_1)$ and $W_2 = span(L_2)$. Show that V is the direct sum of W_1 and W_2 if and only if $L_1 \cup L_2$ is a basis for V.

3. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix}$, and $L_A : \mathbb{R}^3 \to \mathbb{R}^3$,

   (a) Write down a system of linear equations in three variables x, y and z so that x, y and z form a solution to this system of equations if and only if $(x, y, z) \in N(L_A)$.

   (b) Solve this system of equations by using our allowed operations to convert it to an equivalent system in our standard form. Show each step and state what operation you use for each step. Make sure you say what all the solutions of the system are.

   (c) Use your answer to part (b) to find a basis for $N(L_A)$.

   (d) Use the fact that the columns of A are Ae_1, Ae_2, and Ae_3 to find a basis for $R(L_A)$. Be sure to explain how you know this is a basis.

4. Let $T : V \to W$ and $U : W \to Z$ be linear transformations.

   (a) Show that $R(UT) \subseteq R(U)$ and $N(T) \subseteq N(UT)$.

   (b) Give an example in which $N(UT) = N(T)$ and one in which $N(UT) \neq N(T)$.

   (c) Show that if T is surjective then $R(UT) = R(U)$.

   (d) Show that if T is surjective then

   $$nullity(T) + nullity(U) + rank(UT) = dim(V).$$

   (e) Challenge problem (for extra credit only):
   Suppose that $R(T) \cap N(U) = \{0\}$. Find a way to compute the nulity and rank of UT from the nullity and rank of T and U and the dimensions of V, W, and Z. (You should not need to use all of these things.) Prove that your answer is correct.
   Do the same thing if $R(T) \subseteq N(U)$, and if $N(U) \subseteq R(T)$.
   ("Extra credit" means that I will note any extra credit you earn, and take that into account if your final grade at the end of the term is just below the borderline between two letter grades. Don't spend time on this until you have done your best on the rest of the exam, as the rest of the exam is more important to your grade.)