Notes on Heat Equation on a plate

Let Ω be a circle of radius $r = 1$. We want to solve

$$
\begin{aligned}
u_t &= a^2 \Delta u(\boldsymbol{x}) = a^2 \left(u_{rr} + \tfrac{1}{r} u_r\right) \quad 0 \le r \le 1 \\
u(0,t) &= 0 \\
u(r,0) &= F(r)
\end{aligned} \tag{0.1}
$$

We assume that the solution is separable, ie. $u(r,t) = R(r)T(t)$. Plugging this into (0.1) we find

$$RT' = a^2 \left(R'' + \frac{1}{r} R'\right) T$$

We can separate this equation by grouping R's and T's.

$$\frac{T'}{a^2 T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda^2$$

where λ is a constant to be determined.
First solving $\frac{T'}{a^2 T} = -\lambda^2$, we find $T(t) = e^{-a^2 \lambda^2 t}$.
Now we must solve

$$\frac{R'' + \frac{1}{r} R'}{R} = -\lambda^2.$$

Putting everything on one side and multiplying by r^2, we get a second order differential equation

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0 \tag{0.2}$$

Note that this is very similar to the 0^{th} order Bessel equation. To see the difference, lets look for a series solution of the form

$$R(r) = \sum_{n=1}^{\infty} a_n(k) x^{n+k}.$$

Plugging this into (0.2), we find the indicial equation is $k^2 = 0$, $a_1 = 0$ and the recurrence relation for the coefficients is

$$a_n = -\frac{\lambda^2 a_{n-2}}{n^2}.$$

Since $a_1 = 0$, all odd terms must equal zero.

$$a_{2m} = \frac{(-1)^m (\lambda^2)^m a_0}{(m!)^2 2^{2m}}$$

So the series solution is

$$R_1(r) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^m (\lambda r)^{2m}}{(m!)^2 2^{2m}} = J_0(\lambda r).$$

Likewise, the second homogeneous solution is given by $Y_0(\lambda r)$. Thus $R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$. Now Y_0 blows up at the origin so we must set $c_2 = 0$.
Thus $R(r) = c_1 J_0(\lambda r)$. Hence, $u(r,t) = c_1 J_0(\lambda r) e^{-a^2 \lambda^2 t}$.
We know that $u(1,t) = 0 = J_0(\lambda)$. This means that λ must be the roots of J_0. J_0 has infinitely many roots thus by superposition

$$u(r,t) = \sum_{l=0}^{\infty} c_l J_0(\lambda_l r) e^{-a^2 {\lambda_l}^2 t}$$

The initial condition $u(r,0) = F(r)$ determines the coefficients c_l.
The coefficients are (without explanation)

$$c_l = \frac{2}{J_1^2(\lambda_l)} \int_0^1 x J_0(\lambda_l r) F(r) dr.$$