Math 23 Diff Eq: Take-home Midterm

You have until class (10am) on Friday, about 47 hrs. Don't worry; I expect it to take about 4 hours, if you are reasonably prepared. Answer all three questions; try to be clear, concise and neat. You can use the book, the computer (e.g. Matlab), previous HWs, course website resources. If something is unclear, ask me by email/phone/in person. However, as part of the Honor Code, there is no collaboration whatsoever.

1. [20 points] A certain fluid instability is modeled by $y' = y(1 - y^2)$.
   (a) Sketch a phase line, labeling critical points (stating whether stable or unstable).
   (b) In Matlab use the Euler timestepping method with `h=0.1` to produce an approximate plot of the solution in the interval $0 \le t \le 5$, given initial condition $y(0) = 0.01$. Keep a record of your computed value for $y(5)$ [Hint: if `a` is a vector then the last element can be accessed by `a(end)`].
   (c) Overlay plots (using lines) done using a couple of values of `h` each *smaller* than the last by factor 5. Make a little table of $y(5)$ values for each `h` value.
   (d) Stop when you are happy $y(5)$ is accurate to within 1%, and quote its value and your `h`. Explain why you think it's within the desired accuracy.
   (e) Why are so many timesteps (how many?) needed to get a merely adequate 1% accuracy here?

2. [20 points]
   (a) What is the most inclusive t interval where the following is guaranteed to have a unique solution: $(t-2)y'' + \frac{t}{t+2}y' + t(t-2)y = \sin t$, with $y(1) = 1,\ y'(1) = -1$.
   (b) What is the largest radius about $x_0 = 1$ within which the series solution to $(1+x^2)y''+xy'+y = 0$ is guaranteed to converge?
   (c) In quantum physics the Schrodinger equation is very important. Find its general series solution at $x_0 = 0$ in a quadratic potential, that is,

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + 2x^2\right) y = y$$

   Find only the first 3 odd-power terms. For the even-power terms spot the general pattern and write the expression for the n^{th} even term. Bonus: try to recognize this even power series (Hint: substitute $w = x^2$)

3. [25 points] You will now design an automobile suspension system. The body can be modeled by a mass of $m = 500$ kg supported by a single spring of unknown strength (let's not worry about the fact there's 4 wheels).

   (a) In order to react to changes in road height reasonably fast, we want the natural frequency of (undamped) oscillation to be 1.6 cycles per second. Find the needed spring constant k.

   (b) What damping γ is needed to make the system critically damped?

   (c) With this critically-damped system, imagine the road level permanently jumps 10 cm down while driving along. Effectively this means the car is now launched at $t = 0$ from 0.1 m above its (new) equilibrium position, with zero vertical velocity. Analytically solve for the resulting motion. Plot a labeled graph of this solution as displacement vs time.

   (d) Solve, then add to your plot, the motions with the same initial conditions but with γ ten times *bigger* and ten times *smaller* than critically damped. What are the disadvantages of under- and over-damping?

   (e) Plot, or carefully sketch, the motion of the two roots r_1, r_2 in the complex plane, as γ increases from zero up to ∞. What γ value *maximizes* the distance that the *closest* root has to the imaginary axis? (Bonus: connect this to part d).

   (f) On what curve do the roots move as γ increases from zero while remaining underdamped? [Hint: think about the product of roots of a quadratic]. Isn't this cool?