Basic Terminology

An ordinary differential equation (ODE or just DE) is a system with the following ingredients:

- An independent variable (usually t think "time"or x think "position") that derivatives are taken with respect to.
- A dependent variable, i.e. function of the independent variable, e.g. $y = y(t)$ "the variable y which is a function of t".
- A multi-variable function F that describes a relationship between the derivatives of the dependent variable (taken with respect to the independent variable) by

$$F(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \dots, \frac{d^ny}{dt^n}) = 0. \tag{1}$$

For example the equation

$$1 + \sin(y + t) - \left(\frac{y}{t+1}\frac{d^2y}{dt^2}\right)^2 = 0 \tag{2}$$

describes a differential equation with independent variable t and dependent variable y and

$$F(x_0, x_1, x_2, x_3) = 1 + \sin(x_1 + x_0) - (\frac{x_1}{x_0 + 1}x_3)^2.$$

Some comments and definitions

- "Ordinary" implies that there is only one independent variable (i.e. only ever differentiate with respect to that one variable.)
- n is called the **order** of the ODE - $\frac{d^ny}{dt^n}$ is the highest derivative that occurs.
- A **solution** to the ODE is a function (or dependent variable) $y = y(t)$ where the equation (1) holds.

For example, the second order ODE

$$y + \frac{d^2y}{dt^2} = 0 \tag{3}$$

ode has many solutions, for example $y(t) = \sin t$, $y(t) = \cos t$, $y(t) = 2\sin t + \cos t$.

An **initial value problem** (IVP) is an ODE combined with conditions that the solution (and its derivatives) must satisfy at a particular value of the independent variable (e.g. at a point in time). Such as

$$y + \frac{d^2y}{dt^2} = 0, \quad y(0) = 1, \frac{dy}{dt}(0) = 0. \tag{4}$$

Of the given solutions to the ode (3), only $y = \cos t$ is a solution to the IVP (4).

A **boundary value problem** (BVP) is an ODE combined with conditions that the solution must satisfy at various points. Such as

$$y + \frac{d^2y}{dt^2} = 0, \quad y(0) = 1, y(\pi/2) = 2. \tag{5}$$

Here the solution $y = 2\sin t + \cos t$ for the ODE also solves the BVP.

Differential Operators:

An **operator** is a map from functions to functions, i.e. it is a procedure that takes a function and returns another. For example the operator $D = \frac{d}{dt}$ takes a function of t (e.g f=f(t)) and returns its derivative. We write $Df = \frac{df}{dt}$.

A function $p = p(t)$ can be thought of as a **multiplication operator**. Given a function $f = f(t)$, p operates on f by returning the product function pf defined by $(pf)(t) = p(t)f(t)$.

Any two operators can be "multiplied" together. The new action is obtained from the composition of the two operators, e.g.

$$D^2 f = DDf = \frac{d}{dt}\frac{df}{dt} = \frac{d^2 f}{dt^2}.$$

Caution; operator multiplication is not like normal multiplication - **order matters**. For example if $p = t^2$ then $(pD)f = t^2 \frac{df}{dt}$ but $(Dp)f = \frac{d}{dt}(t^2 f) = t^2 \frac{df}{dt} + 2tf$.

A **differential operator** is a combination of D, multiplication operators and familiar operations such as addition and multiplication. It is often useful to write ODE's in terms of operators, e.g. (3) can be re-written as $(D^2 + 1)y = 0$. Here D^2 just means differentiate twice (with respect to the independent variable) and 1 is the multiplication operator that takes a function and multiplies it by 1 (i.e. returns the same function). The multiplication operator 1 is known as the identity operator. As another example the ODE

$$1 + t^2 y + \cos t \frac{dy}{dt} = e^t \tag{6}$$

can be written as

$$(t^2 + \cos t D)y = e^t - 1. \tag{7}$$

The left consists of a differential operator applied to the dependent variable, the right is a function of the independent variable alone.

An operator P is linear if $P(f+g) = Pf + Pg$ for any functions f, g and $P(af) = aPf$ for any constant a. All multiplication operators and powers of D are linear. For example

$$D^2(f+g) = \frac{d^2}{dt^2}(f(t) + g(t)) = \frac{d^2 f}{dt^2} + \frac{d^2 g}{dt^2}.$$

But the operator $f \mapsto (\frac{df}{dt} + 1)^2$ is not linear as $af \mapsto (a\frac{df}{dt} + 1)^2 \neq a(\frac{df}{dt} + 1)^2$.

An ODE is linear if it can be written in the form

$$Py = g(t)$$

where P is a linear differential operator and $g(t)$ is some function of t. For example the ODEs (6) and (3) are linear but the ODE (2) is not. Anything involving higher powers of y or its derivatives, for example, will be nonlinear.

A linear ODE is **homogeneous** if $g(t) = 0$ and **inhomogeneous** otherwise.