Math 22 Summer 2014 Midterm Exam I

Friday July 11, 2014

PRINT NAME: Commented Solutions

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed: A wrong guess does not cost you more points than leaving the question unanswered.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

Warning: the echelon form is non-unique. My echelon forms might be different from yours, but the pivots are in the same place.

1. (a) [10 pt] Solve the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 9x_3 & + & 2x_4 & = & -10 \\ -3x_1 & + & x_2 & + & 30x_3 & - & 7x_4 & = & 36 \\ 2x_1 & + & 3x_2 & - & 9x_3 & + & 2x_4 & = & -9 \end{array}$$

(b) [3 pt] Write the solution you found in parametric vector form, if possible.

(a)

$$\begin{pmatrix} 1 & 0 & -9 & 2 & -10 \\ -3 & 1 & 30 & -7 & 36 \\ 2 & 3 & -9 & 2 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -9 & 2 & -10 \\ 0 & 1 & 3 & -1 & 6 \\ 0 & 3 & 9 & -2 & 11 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & -9 & 2 & -10 \\ 0 & 1 & 3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \end{pmatrix}$$

$$\begin{cases} x_1 = 9x_3 + 4 \\ x_2 = -3x_3 - 1 \\ x_3 \text{ is free} \\ x_4 = -7 \end{cases}$$

Remember to specify that x_3 is a free variable.

(b)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9x_3 + 4 \\ -3x_3 - 1 \\ x_3 \\ -7 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} 9 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

2. Let T be the linear transformation given by

$$T(x_1, x_2, x_3) = (2x_2 + 10x_3,\ x_3,\ x_1 + 4x_2 + 5x_3,\ 2x_2 + 3x_3)$$

(a) [3 pt] What's the standard representing matrix for T?

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 + 10x_3 \\ x_3 \\ x_1 + 4x_2 + 5x_3 \\ 2x_2 + 3x_3 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 2 & 10 \\ 0 & 0 & 1 \\ 1 & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$$

Note that doing row operations on A changes the transformation T. So the correct answer is the matrix A as is.

(b) [2 pt] What's the domain of T? What's the codomain of T?

Since A is 4×3 the domain is $\mathbb{R}^3$, the codomain is $\mathbb{R}^4$. This should also be clear from how T has been written in column in part (a): it takes in a vector with 3 entries and gives a vector with four entries.

(c) [5 pt] Is T onto? Is T one-to-one Why or why not?

Here we row-reduce A. Because it's a coefficient matrix there is no issue of consistency or inconsistency.
T is onto if A has a pivot in every row, and it's one-to-one if it has a pivot in every column

$$A \sim \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 10 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 4 & 5 \\ 0 & \boxed{2} & 10 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

pivot in every column $\Rightarrow$ T is one-to-one

last row has no pivot. $\Rightarrow$ T is not onto.

Remember that a proper echelon form must have zeroes below the pivots. If that's not the case the process is incomplete, and I took off a point for it.

3. Consider the following matrix

$$A = \begin{pmatrix} 4 & -3 & -6 \\ -7 & 6 & 9 \\ -1 & 1 & 1 \end{pmatrix}$$

(a) [8 pt] Write the solution to the homogeneous system $Ax = 0$ *as the span of a set of vectors.*

$$A \sim \begin{pmatrix} -1 & 1 & 1 \\ 4 & -3 & -6 \\ -7 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} +1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 3x_3 \\ x_2 = 2x_3 \\ x_3 \text{ is free} \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \text{The solution set is } \mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

To write the solution of the homogeneous system as a span first we write the solution ~~as~~ in parametric vector form, then take those vectors. If the solution had been $x = x_2 u + x_4 v + x_5 w$, the way to write it as a Span is $\mathrm{Span}\{u, v, w\}$.

(b) [2 pt] Is the solution set of $Ax = 0$ a point, line or plane? Explain your answer.

Span of one nonzero vector is a line. Try to be precise when justifying answers.

(c) [2 pt] Are the columns of A linearly independent? Why or why not?

No, they are not LI because the homogeneous system has a nontrivial solution.

(d) [3 pt] Given that

$$\begin{pmatrix} 4 & -3 & -6 \\ -7 & 6 & 9 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ -17 \\ -2 \end{pmatrix}.$$

find all the solutions to the equation $Ax = b$ with $b = \begin{pmatrix} 11 \\ -17 \\ -2 \end{pmatrix}$ **without solving the linear system** $(A \quad b)$.

This is the question most people got wrong. You shoud notice that it's part (d) of question 3, and that the equation on top of the page says $A\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = b$, i.e. that $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ is a solution to $Ax = b$.

Because the matrix A is the same as in part (a) you know that $Ax = 0$ has solution set $V_h = \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right\}$.

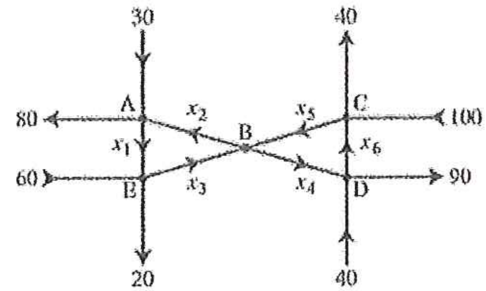
Then you use the fact that the solution set to $Ax = b$ is equal to a particular solution (i.e. a solution to $Ax = b$) plus the solution set to $Ax = 0$.

All in all, the solutions to $Ax = \begin{pmatrix} 11 \\ -17 \\ -2 \end{pmatrix}$ are given by

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right\}$ or, equivalently $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

4. [10 pt]

Set up and **DO NOT SOLVE** the linear system associated to the following network flow.



Total contribution entering a node = Total contribution leaving ~~the node~~ the node and total flow entering the network = total flow leaving the network.

	IN		OUT
Network:	$30+60+40+100$	$=$	$40+90+80+20$
A	x_2+30	$=$	x_1+80
B	x_3+x_5	$=$	x_2+x_4
C	x_6+100	$=$	x_5+40
D	x_4+40	$=$	x_6+90
E	x_1+60	$=$	x_3+20

5. [15 pt] Given the vectors v_1, v_2, v_3, b as follows

$$v_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ -4 \\ 5 \\ 7 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5 \\ -3 \\ 6 \\ 5 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -7 \\ -1 \\ -2 \end{pmatrix}$$

is b in the span of the set $\{v_1, v_2, v_3\}$?

b is in the span if and only if $(v_1 \; v_2 \; v_3 \; b)$ is consistent.

$$\begin{pmatrix} 1 & 4 & 5 & -1 \\ -3 & -4 & -3 & -7 \\ 2 & 5 & 6 & -1 \\ 3 & 7 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 & -1 \\ 0 & 8 & 12 & -10 \\ 0 & -3 & -4 & 1 \\ 0 & -5 & -10 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 & -1 \\ 0 & 4 & 6 & -5 \\ 0 & -3 & -4 & 1 \\ 0 & -5 & -10 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 4 & 5 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & -3 & -4 & 1 \\ 0 & -5 & -10 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 2 & -11 \\ 0 & 0 & 0 & -19 \end{pmatrix}$$

System is inconsistent because of last row

$\Rightarrow$ b is not in the Span$\{v_1, v_2, v_3\}$

6. [7 pt] Find all vectors $b \in \mathbb{R}^2$ such that the system $Ax = b$ has a solution, where A is the following matrix:

$$A = \begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix}$$

We're trying to find $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ so that $(A \; b)$ is consistent.

$$\begin{pmatrix} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & b_1 \\ 0 & 0 & b_2+3b_1 \end{pmatrix}.$$

For the system to be consistent the last row must always be zero, so $b_2 = -3b_1$ and ~~all~~ the vectors we're looking for are $b = b_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

It was enough to recognize that $b_2 + 3b_1 = 0$.

7. **Multiple choice section:** each question is worth 5 points. No justification is needed, so don't leave any question blank.

(a) Consider the following transformation

$$T(x_1, x_2, x_3) = (x_3^2 - 5x_1,\ x_3 + 3x_1 - x_2,\ 0,\ 4x_3 - 2x_2).$$

Use the vectors in one the following sets (and ONLY those vectors) you can show that T is **NOT** a linear map. Which set is that?

$$(A)\ \begin{pmatrix}0\\0\\0\end{pmatrix} \quad (B)\ \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}2\\6\\0\end{pmatrix} \quad (C)\ \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$(D)\ \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix} \quad (E)\ \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\\-4\end{pmatrix},\quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\\4\end{pmatrix} \neq -\begin{pmatrix}1\\-1\\0\\-4\end{pmatrix}$$

The issue is the $(x_3)^2$ in the first component.

ANSWER $\boxed{D}$

(b) Only one of the following sets of vectors is made of linearly independent vectors. Find that set.

$$(A)\ \begin{pmatrix}1\\5\\7\end{pmatrix}, \begin{pmatrix}0\\2\\5\end{pmatrix}, \begin{pmatrix}7\\-3\\-2\end{pmatrix}, \begin{pmatrix}-15\\3\\0\end{pmatrix} \quad (B)\ \begin{pmatrix}-3\\2\\3\end{pmatrix} \begin{pmatrix}2\\-\frac{4}{3}\\-1\end{pmatrix} \quad (C)\ \begin{pmatrix}5\\17\\-3\end{pmatrix} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$(D)\ \begin{pmatrix}2\\9\end{pmatrix} \begin{pmatrix}-1\\-7\end{pmatrix} \begin{pmatrix}5\\9\end{pmatrix} \begin{pmatrix}-3\\-4\end{pmatrix} \quad (E)\ \begin{pmatrix}-2\\6\\-14\end{pmatrix} \begin{pmatrix}1\\-3\\7\end{pmatrix}$$

A, D have more vectors than entries,
C has the zero vector
and in E $\begin{pmatrix}-2\\6\\-14\end{pmatrix} = -2\cdot\begin{pmatrix}1\\-3\\7\end{pmatrix}$, so all those sets are LD.

ANSWER $\boxed{B}$

(c) Suppose that

$$u = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}, v = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

and that you know that $2u - w = 3v$. Using this fact find $x_1, x_2 \in \mathbb{R}$ that satisfy the equation

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

(A) $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ (B) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (C) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (E) $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

ANSWER

$\boxed{E}$

Note the matrix equation is $(u, v)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = w \Rightarrow x_1 u + x_2 v = w$,

and $2u - w = 3v \Rightarrow 2u - 3v = w$

(d) Suppose that you have a linear transformation T whose standard matrix has the following echelon form:

$$A = \begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \end{pmatrix}$$

where $\bullet$ corresponds to a pivot position. Which of the followng statements is true?

(A) T is one-to-one and onto; (B) T is one-to-one but not onto;

(C) T is onto but not one-to-one; (D) T is neither onto nor one-to-one;

ANSWER

$\boxed{C}$

See question 2 (c).

TRUE or FALSE?

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. For each incorrect statement you'll lose 2 points. If five or more statements are incorrect you'll get 0 points out of the question.

8. [10 pt]

(a) **True** / **False** A system of 3 equations in 4 unknowns always has a solution.
Could be inconsistent, see $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 = 2 \\ 0 = 1 \end{cases}$

(b) **True** / **False** $Ax = b$ always has a solution if there is a pivot position in every row of A.

(c) **True** / **False** The free variables in the solution of a linear system correspond to the pivot columns.
They correspond to the non-pivot columns

(d) **True** / **False** If a system of linear equations has two solutions there must be infinitely many.
Either zero, one or infinitely many.

(e) **True** / **False** A linear transformation T is onto if all rows of its standard matrix A have a pivot position.

(f) **True** / **False** A set of one vector is always linearly independent. $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ is linearly dependent.

(g) **True** / **False** A system $Ax = b$ is consistent if the last column of the augmented matrix $(A \;\; b)$ is a pivot column.
If the last column of $(A\ b)$ is a pivot then the last row is $(0 \cdots 0 *)$, which is an inconsistency row

(h) **True** / **False** The solution set to $Ax = b$ is $p + v_h$, where p is a solution and v_h is the general solution of $Ax = 0$.
See 3 (d).

(i) **True** / **False** The matrix-vector product Ax is the linear combination of the columns of A with weights given by the entries of x.

(j) **True** / **False** A linear transformation T is one-to-one if all rows of its standard matrix A have a pivot position.
It's one-to-one if all columns have a pivot.