Math 22, Summer 2013, Final Exam

Name (Print): Solutions

Last First

Instructions: You are not allowed to use calculators, books, or notes of any kind. You may not look at a classmate's exam for "inspiration." You must explain your reasoning behind each solution to receive full credit. Credit will not be awarded for correct answers with no explanation (with the exception of problem #1).

You may use pages 14-17 of the exam as scratch paper, but any work you intend to be graded should be on the exam itself in the space provided. If you run out of room, clearly indicate which page of scratch paper your solution is on and circle the solution that should be graded.

Before beginning the exam, skim through the problems to verify that you have one true/false question and nine free-response questions.

1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):

   (a) (True / False) Suppose A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$. Then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   (b) (True / False) If A is an $n \times n$ symmetric matrix, then the dimensions of the eigenspaces of A must sum to n.

   (c) (True / False) Suppose A is a 3×5 matrix. Then the columns of A are linearly independent.

   (d) (True / False) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

   (e) (True / False) If A is row equivalent to B, then Col$A$=Col$B$.

Explanations

(a) For $Ax=b$ to be inconsistent, the last row of $(A|b)$ must be equivalent to $(0 \cdots 0\ x)$, so A has no pivot in the last row. But it's $n \times n$, which means it must have a free variable ($n-1$ pivots at most).

(b) Spectral theorem

(c) Since A is 3×5 it can have at most 3 pivots $\Rightarrow$ there is a column w/o pivot $\Rightarrow$ columns are LD

(d) Invertible matrix theorem

(e) While the same columns of A and B span Col(A) and Col(B) resp, the spaces are different

2. Consider the set $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \right\}$

(a) Is S linearly independent? Explain.

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 18 \\ 0 & -1 & 18 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 33 \\ 0 & 1 & -18 \\ 0 & 0 & 0 \end{pmatrix}$$

Column w/o pivot $\Rightarrow$ columns are LD

3. For what value(s) of h will $\mathbf{y}$ be in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

It happens if $(v_1\ v_2\ v_3\ y)$ is consistent

$$\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix} \implies$$ system is consistent if $h = 5$

4. (a) Determine the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects the points in $\mathbb{R}^2$ across the x_1-axis, and then reflects the points across the line $x_2 = x_1$. Show your work. (2pts)

Note: I would give you the matrices in the problem.

Reflection along x_1 : $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Reflection along $x_2 = x_1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$[T] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) Draw a possible echelon form for the standard matrix of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is onto, using '□' to represent pivot positions, '0' for entries that must be zero, and '*' for the remaining entries. (1pt)

$T : \mathbb{R}^4 \to \mathbb{R}^3$ is a 3×4 matrix. Onto if all rows have a pivot.

$$\begin{pmatrix} \square & \times & \times & \times \\ 0 & \square & \times & \times \\ 0 & 0 & \square & \times \end{pmatrix} \text{ or } \begin{pmatrix} \square & \times & \times & \times \\ 0 & 0 & \square & \times \\ 0 & 0 & 0 & \square \end{pmatrix}$$ would work

(c) Can T in part (b) be one-to-one? Why or why not? (2pts)

No, T would need a pivot in every column, and it can have at most 3 pivots.

5. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is closed under vector addition. [Hint: if $\mathbf{u}$ and $\mathbf{v}$ are vectors in the range of T, this implies that there exist vectors $\mathbf{x}$ and $\mathbf{y}$ in the domain V of T such that $\mathbf{u} = T(\mathbf{x})$ and $\mathbf{v} = T(\mathbf{y})$].

Nct

Let u, v be in the range, so that $u = T(x)$ and $v = T(y)$ for some x, y in the domain

Then $u + v = T(x) + T(y) = T(x+y)$ is also in the range.

6. (a) If A is a 5×9 matrix, what is the smallest possible dimension of Nul$A$? Explain.

A can have at most 5 pivots, so A has at least four free variables (i.e. columns w/o pivot).
So dim(Nul(A)) is at least 4

(b) A scientist solves a nonhomogeneous system of 8 linear equations in 10 unknowns and finds that 4 of the unknowns are free variables. Can this scientist be certain that if the right hand side of the equations are changed, the new nonhomogeneous system will have a solution? Explain.

Use Rank-Nullity thm: rank(A) + dim(Nul(A)) = # columns

$\Rightarrow$ rank(A) = 10 - 4 = 6. Because A has 8 rows, there are rows w/o pivot, so for some choices of the RHS there will be no solution

(c) Suppose the matrix A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Determine rank$A$, dimNul$A$, dimCol$A$, and dimRow$A$. NCX

rank(A) = 2 = dim(Col(A)) = dim(Row(A))
dim(Nul(A)) = 3

7. Consider the set $\mathcal{B} = \{p_1, p_2, p_3\}$, where $p_1(t) = 1-3t^2$, $p_2(t) = 2+t-5t^2$, and $p_3(t) = -1-2t+t^2$. Let $\mathcal{E} = \{1, t, t^2\}$, the standard basis for $\mathbb{P}_2$. Using the fact that the linear transformation $[\ \ ]_{\mathcal{E}}$ is an isomorphism from the vector space $\mathbb{P}_2$ onto the vector space $\mathbb{R}^3$, determine whether $\mathcal{B}$ is a linearly independent or a linearly dependent set in $\mathbb{P}_2$.

NGt

The std matrix is $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ -3 & -5 & 1 \end{pmatrix}$

Row-reduce it

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ -3 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

No pivot in last column $\Rightarrow$ LD

8. Let $\mathbf{y} = \begin{bmatrix} 12 \\ 14 \\ 25 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$. Let $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. (1pt)

$$u_1 \cdot u_2 = -2 + 6 - 4 = 0$$

(b) Write $\mathbf{y}$ as $\hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$, where $\hat{\mathbf{y}}_1 \in W$ and $\hat{\mathbf{y}}_2 \in W^{\perp}$. (4pts)

$$\hat{y}_1 = \text{proj}_{u_1} y + \text{proj}_{u_2} y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 =$$

$$= \frac{27}{9} u_1 + \frac{130}{26} u_2 = 3u_1 + 5u_2 = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 15 \\ 20 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 21 \\ 17 \end{pmatrix}$$

$$\hat{y}_2 = y - \hat{y}_1 = \begin{pmatrix} 12 \\ 14 \\ 25 \end{pmatrix} - \begin{pmatrix} 1 \\ 21 \\ 17 \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \\ 8 \end{pmatrix}$$

9. Let $A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \\ -4 & 4 \end{bmatrix}$. The eigenvalues of A^TA are $\lambda_1 = 48$ and $\lambda_2 = 32$ (Note: $48 = 16 \cdot 3$ and $32 = 16 \cdot 2$). Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\{\mathbf{v}_i\}$ is a basis for the eigenspace of A^TA associated with λ_i, $i = 1, 2$. Suppose $A = U\Sigma V^T$ is a singular value decomposition of A.

(a) Determine V. (1pt)

Since v_1, v_2 are orthonormal, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) Determine Σ. (2pt)

$$\Sigma = \begin{pmatrix} \sqrt{48} & 0 \\ 0 & \sqrt{32} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D \\ 0 \end{pmatrix}$$

(c) Determine U. (2pt)

$$Av_1 = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \qquad u_1 = \frac{Av_1}{\sqrt{48}} = \begin{pmatrix} 4/\sqrt{48} \\ 4/\sqrt{48} \\ 4/\sqrt{48} \end{pmatrix}$$

$$Av_2 = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} \qquad u_2 = \frac{Av_2}{\sqrt{32}} = \begin{pmatrix} 4/\sqrt{32} \\ 0 \\ -4/\sqrt{32} \end{pmatrix}$$

u_3 is a unit vector that forms a basis of $\mathrm{Nul}(A^T)$

$$A^T = \begin{pmatrix} 4 & 0 & -4 \\ 4 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & -4 \\ 0 & 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow z = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$

10. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the subspace W of $\mathbb{R}^3$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Use the Gram-Schmidt process to determine an orthogonal basis for W.

NEX

$$z_1 = v_1$$

$$z_2 = v_2 - \frac{v_2 \cdot z_1}{z_1 \cdot z_1} z_1 = v_2 - \frac{2}{3} v_1 =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \\ 4/3 \end{pmatrix}$$